THESES

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By

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Ist Thesis:- On the S-Operator associated with the Lie group of transformations.

2nd Thesis:- On the study of the Lorentz invariant spaces.

3rd Thesis:- On the geometry of the spaces of Riemann constructed by representing hyperquadrics as points.

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A C K N O W L E D G M E N T.

It

~~This~~ is a pleasant duty for me, at the beginning of this work, to thank very respectfully my respected teachers Professor D.D.Kosambi, Professor M.R.Siddiqi and Professor Father C. Racine for their trainings, suggestions and encouragements which have been very precious to me. I also owe much to the works of Professor E.Cartan of Paris which I had occasion to consult during this work. in the course of

FIRST THESIS.

On the S-Operator associated with Lie group of transformations.

INTRODUCTION.

The deformation of a space V_n, whether metric or of affine connexion, is realised by remarking that a change of variables

$$\bar{x}^i = f^i(x),$$

admits two different geometrical interpretations. Kosambi has pointed out in his paper " Continuous groups and two theorems of Euler, Math. Student, II, 1934, pp. 94-100" that the change of variables can be regarded either (a) as a mapping of the x-space upon the $\bar{x}$-space, i.e. as a straightforward transformation of the x-space into the $\bar{x}$-space, or (b) as a mapping of the x-space upon itself, i.e. the point of coordinates $\bar{x}^i$ is regarded as the point of coordinates $\bar{x}$ in the x-space. The change of variables, in the second interpretation, will be called a displacement, i.e. the point x is displaced in the original space V_n to $\bar{x}$. Thus in the displaced V_n vectors and tensors at $\bar{x}$ are just the vectors and tensors of V_n at $\bar{x}$, and the metric and connexion parameters are $g_{ij}(\bar{x})$ and $L^i_{jk}(\bar{x})$ This displaced space will also be referred to as the deform of V_n. Introducing, then, the notations $\bar{g}_{ij}(\bar{x})$ and $\bar{L}^i_{jk}(\bar{x})$ for the metric and connexion parameters of the transformed space V_n at $\bar{x}$, we consider $\bar{g}_{ij}(\bar{x})$ and $\bar{L}^i_{jk}(\bar{x})$ as the representatives of V_n at the point $\bar{x}$. The measure of deformation of V_n can be obtained by comparing the metric and connexion parameters of the new geometry of the deformed V_n at $\bar{x}$, with the representatives of V_n at $\bar{x}$.

In my paper " one parameter continuous group of deformations" I have shown that by subjecting a general space of affine connexion to general finite one parameter groups of transformations G_1, defined by

$$\bar{x}^i = x^i + tDx^i + \ldots\ldots + \frac{t^n}{n!} D^n x^i + \ldots$$

where t is a parameter, and $D = u^r \frac{\partial}{\partial x^r}$, u^r being the vector field defining the infinitesimal transformation that generates the group, the deformations of the vectors, tensors and the connexion parameters (L^i_{jk}) of the space are determined by a single operator, which I call S-operator associated with the given one-parameter group of Lie.

It is to be noticed that the exposition of this problem though given in a different manner in my paper cited above as well as in Chapter II of this work, is the same at the foundation as that given above.

Chapter I of this work has its object to give a brief exposition of the method of '<u>Repère mobile</u>' as initiated by Darboux and developped, in all its generalities, by E. Cartan to study the invariants of a variety and the generalizations of Riemann spaces. The method of '<u>Repère mobile</u>' was later on generalized by E. Cartan to establish the structure of the finite continuous groups and to make the connection of his theory with the classical theory of S. Lie, because the elements of the theory of E. Cartan are found in the notions introduced by S. Lie.

In Chapter II, I give a short proof of my results about the S-operator which I have already defined in my paper to be published shortly in Ind. Jour. Math., by the methods of tensor calculus. The present method of defining the S-operator is based on the theory of '<u>Repère mobile</u>' and was very kindly indicated to me by Father C. Racine.

I give here, once for all, some formulae of '<u>Covariant bilinear</u>' introduced by Frobenius and '<u>Exterior multiplication</u>' defined by H. Grassmann as they will be constantly applied in the present work.

'Covariant bilinear'. - If

$$\omega(x, dx) = \sum_K a_K(x)\, dx_K$$

be a form of Pfaff,

the expression

$$d\omega(x, \delta x) - \delta\omega(x, dx) = \sum_K \left(d a_K \delta x_K - \delta a_K dx_K\right)$$

$$= \sum_{(iK)} \left(\frac{\partial a_K}{\partial x_i} - \frac{\partial a_i}{\partial x_K}\right)\left(dx_i \delta x_K - \delta x_i dx_K\right)$$

$$(1 \leq i \leq K \leq n)$$

will be denoted by an abridged symbol

$$\boxed{\omega'(x, dx, \delta x) = d\omega(x, \delta x) - \delta\omega(x, dx)}$$

'<u>Exterior multiplication</u>' - The 'exterior' product of p differentials $dx_1, dx_2, \ldots, dx_p$ is denoted by

$$[dx_1\, dx_2 \ldots dx_p]$$

and stands for the determinant

$$\begin{vmatrix} \delta_1 x_1 & \delta_2 x_1 & \cdots & \delta_p x_1 \\ \delta_1 x_2 & \delta_2 x_2 & \cdots & \delta_p x_2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \delta_1 x_p & \delta_2 x_p & \cdots & \delta_p x_p \end{vmatrix}$$

what are $\delta_1, \ldots, \delta_p$?

The 'exterior' product of the forms

$$\omega_1 = \sum a_i dx_i \;,\; \omega_2 = \sum b_i dx_i \;,\; \omega_3 = \sum C_i dx_i$$

will be by definition

$$[\omega_1 \omega_2 \omega_3] = \sum_{i,j,K} a_i b_j C_K [dx_i dx_j dx_K] .$$

CHAPTER I.

Method of Repère Mobile.

Trièdre mobile: equations of Darboux.

1. In his treatise " Leçons Sur la théorie général des surfaces" Darboux introduced an extremely fruitful method of rectangular '*trièdre mobile*' in the study of curves and surfaces in euclidean spaces. This was, in reality, to recall the properties of the group of displacements of the space. Moreover, as H. Poincaré remarked, the notion of the group of displacements plays a fundamental rôle in geometry: the equality of two figures is defined by the possibility of superposing them, that is to say, to transform one figure into the other by a displacement. For fixing the ideas, let us take the group of displacements of the space. This group of displacements depends on six parameters. If we take a rectangular '*trièdre*' (T_0) and apply to it a displacement S_a, the '*trièdre*' (T_a) thus obtained can be regarded as a system of reference. The relative coordinates of a point referred to (T_a) are deduced from its absolute coordinates referred to (T_0) by the analytic transformation S_a^{-1}. Consider two infinitely near

'trièdres' (T_a) and (T_{a+da}); the passage of the one to the other is made by the infinitesimal displacement $S_{a+da} S_a^{-1}$, this displacement being the product of two successive displacements S_a^{-1} and S_{a+da}. To define this displacement analytically by using the relative coordinates with respect to the '<u>trièdre</u>' (T_a), it is necessary for us to replace the geometrical transformation $S_{a+da} S_a^{-1}$ by the analytic transformation given by

$$S_a^{-1} (S_{a+da} S_a^{-1}) S_a = S_a^{-1} S_{a+da}$$

Its symbol is of the type

$$\boxed{\begin{aligned} S_a^{-1} S_{a+da} &= \omega_1(a,da) X_1 f + \omega_2 (a,da) X_2 f + \omega_3 X_3 f \\ &\quad + \omega_{23} X_{23} f + \omega_{31} X_{31} f + \omega_{12} X_{12} f \end{aligned}}$$

where $X_1 f$, $X_2 f$, $X_3 f$ are the infinitesimal translations parallel to the axes of coordinates, $X_{23} f$, $X_{31} f$, $X_{12} f$ are the infinitesimal rotations round the axes of coordinates and $\omega_1, \omega_2, \omega_3, \omega_{23}, \omega_{31}, \omega_{12}$ are the six forms of Pfaff of the six parameters $a_1, \ldots\ldots, a_6$ of the group of displacements.

These expressions of Pfaff are called the six relative components of the infinitesimal displacement of the '<u>trièdre mobile</u>'. The relative components of the infinitesimal displacement are defined by the following relations:-

$$
\begin{aligned}
\overrightarrow{dP} &= \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3 ,\\
\overrightarrow{de}_1 &= \qquad\qquad \omega_{12} \vec{e}_2 - \omega_{31} \vec{e}_3 ,\\
\overrightarrow{de}_2 &= -\omega_{12} \vec{e}_1 \qquad\qquad + \omega_{23} \vec{e}_3 ,\\
\overrightarrow{de}_3 &= \omega_{31} \vec{e}_1 - \omega_{23} \vec{e}_2
\end{aligned}
$$

where $[\vec{e}_1, \vec{e}_2, \vec{e}_3]$ are the three unit vectors along the edges of the '<u>triѐdre mobile</u>' (T) at the point P.

2. Let a '<u>triѐdre mobile</u>' depend on two parameters u and v. In this case the relative components of the translation and the rotation are given by

$$\omega_1 = \xi du + \xi_1 dv, \quad \omega_2 = \eta du + \eta_1 dv, \quad \omega_3 = \zeta du + \zeta_1 dv,$$

$$\omega_{23} = p du + p_1 dv, \quad \omega_{31} = q du + q_1 dv, \quad \omega_{12} = r du + r_1 dv,$$

where $\xi, \xi_1, \ldots\ldots, r, r_1$, are functions of u and v.

These functions are not arbitrary functions of the parameters u and v but are subjected to the <u>only</u> relations, for the first time established by Darboux, given by a system of differential equations

$$\frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = (q\zeta_1 - q_1\zeta) - (r\eta_1 - r_1\eta),$$

$$\frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = (r\xi_1 - r_1\xi) - (p\zeta_1 - p_1\zeta),$$

$$\frac{\partial \zeta}{\partial v} - \frac{\partial \zeta_1}{\partial u} = (p\eta_1 - p_1\eta) - (q\xi_1 - q_1\xi),$$

$$\frac{\partial p}{\partial v} - \frac{\partial p_1}{\partial u} = q r_1 - q_1 r, \quad \frac{\partial q}{\partial v} - \frac{\partial q_1}{\partial u} = r p_1 - r_1 p,$$

$$\frac{\partial r}{\partial v} - \frac{\partial r_1}{\partial u} = p q_1 - p_1 q .$$

These equations are called the classical equations of Darboux and contain in them all the differential geometries of the euclidean space. For this reason the chief importance of the method of '<u>trièdre mobile</u>' lies in the fact that all the intrinsic properties of the euclidean space are contained in the relative components of the infinitesimal displacement of the '<u>trièdre mobile</u>' attached to a point of the space.

<u>Generalization of the notion of 'trièdre mobile'</u>.

3. In a space of n dimensions $(x_1, \ldots, x_n)$ we consider a continuous group G of order r which plays with respect to the geometry of this space the same rôle as the group of displacements plays with respect to the euclidean space.

We can consider, in a similar manner as in the euclidean space, a family of '<u>Repère</u>' (R) such that any two of them can be brought into coincidence by one and only one transformation of the group G. With each <u>repère</u> we associate a system of coordinates deduced from a given primitive system by a transformation of G. The infinitesimal transformation of the group which brings two infinitely near <u>repères</u> into coincidence possesses the relative components $\omega_i(a, da)$ with respect to the coordinates associated with R_a. These relative components satisfy the equations of structure of the group G and contain in them the intrinsic properties of the variety based on the given group G. This puts into evidence the rôle which the method of '<u>Repère mobile</u>' plays in the investigation of the differential invariants of a variety.

<u>General definition of the repère mobile.</u>

4. The transformations of a continuous group G of order r are defined by the equations of the form

$$x_i' = \varphi_i(x_1, \ldots, x_n ; a_1, \ldots, a_r) \qquad (i = 1, \ldots, n)$$

where $\dfrac{D(\varphi_1, \ldots, \varphi_n)}{D(x_1, \ldots, x_n)} \neq 0.$

These equations define a system of reference, R_0, on a domain D. Let the family of finite continuous transformations S_a, constituting the group G_r, operate on the same domain D and let R_0 be the absolute '<u>repère</u>' of D. Let us trace a figure F_0 in D and associate the figure S_aF_0 to the system of reference $S_aR_0 = R_a$. There must correspond two distinct figures S_aF_0 and S_bF_0 to two distinct systems of reference $R_a = S_aR_0$ and $R_b = S_bR_0$. F_0 must differ from all the figures $S_a^{-1}S_bF_0$.

For example, we can constitute the figure F_0 with a finite number of points $A_1, A_2, \ldots\ldots, A_p$: A_1 will be a point which is not invariant by the transformations $S_a^{-1}S_b$. Let Σ_1 be those of the transformations $S_a^{-1}S_b$ which leave A_1 invariant; A_2 will be a point which is not identical with all of its transformed points by Σ_1 ; let Σ_2 be those of the transformations $S_a^{-1} S_b$ which leave A_1 and A_2 invariants, and so on. The last point will be such that Σ_{p+1} is reduced to an identity transformation Thus F_0 must not be invariant by any transformation of the group other than the identity.

Having chosen such a figure F_0, distinct from all of its transformed figures $S_a^{-1} S_b F_0$, we shall say that the figure $S_a F_0$ is the <u>repère</u> attached to the system of reference $R_a = S_a R_0$. Instead of saying ' a system of reference' we shall hereafter use the term ' <u>the corresponding repère</u>'.

<u>Relative and absolute components of a repère mobile.</u>

5. Consider the '<u>repère mobile</u>' of a family of transformations of a group G. If R_a and R_{a+da} be two infinitely near positions of this <u>repère</u>, the geometrical transformation which transforms R_a into R_{a+da} is given by $S_{a+da} S_a^{-1}$. Referred to R_a it is represented by the analytic transformation
$S_a^{-1} (S_{a+da} S_a^{-1}) S_a = S_a^{-1} S_{a+da}$.

<u>Definition</u>: We say that a 'repère mobile' R_a possesses the relative components $\omega_i(a, da)$ when the symbol of the infinitesimal transformation $S_a^{-1} S_{a+da}$ is of the type

$$\boxed{S_a^{-1} S_{a+da} = \omega_1 X_1 f + \cdots + \omega_r X_r f}$$

the infinitesimal transformations $X_1 f, \ldots, X_r f$ do not depend on the parameters $a_1, \ldots, a_r$ and their differentials $da_1, \ldots, da_r$; $\omega_1(a,da), \ldots, \omega_r(a,da)$ are the r forms of Pfaff. At any point of the space of parameters these forms are in general independent.

In order that the relative components of a '*repère mobile*' exist, it is necessary and sufficient that

$$\boxed{\delta x_i = \sum_K^{1,\ldots,r} \omega_K(a,da) X_K(x_i)}$$

Definition: We say that R_a possesses absolute components when the symbol of the analytic transformation $S_{a+da} S_a^{-1}$, which is referred to R_0 and which transforms R_a into R_{a+da}, can be written as

$$\boxed{S_{a+da} S_a^{-1} = \tilde{\omega}_1(a,da) Y_1 f + \cdots + \tilde{\omega}_r(a,da) Y_r f}$$

where $Y_1 f, \ldots, Y_r f$ are the infinitesimal transformations independent of $a_1, \ldots, a_r$ and $da_1, \ldots, da_r$; $\tilde{\omega}_1, \ldots, \tilde{\omega}_r$ are the r forms of Pfaff which are in general independent at any arbitrary point of the space of parameters.

In order that the absolute components of a '*repère mobile*' exist; it is necessary and sufficient that

$$\boxed{\delta x_i = \sum_K^{1,\ldots,r} \tilde{\omega}_K(a,da) Y_K(x_i)}$$

Relative components of an affine repère.

6. Let us consider the affine group in a space of n dimensions. We suppose that the parameters are chosen in such a way that the identity transformation is obtained by putting all the parameters to zero.

Let us find out the components of the infinitesimal transformation S_{da}. The repère R_{da} is composed of a point P and n vectors $[\vec{e}_1, \ldots\ldots, \vec{e}_n]$. In order to locate it, it is sufficient to give the coordinates $\omega_1(o,da), \omega_2(o,da), \ldots, \omega_n(o,da)$ of P, with respect to the repère R_0, and the components $1+\omega_{11}(o,da), \omega_{12}(o,da), \ldots, \omega_{1n}(o,da)$ of $\vec{e}_1$, the components $\omega_{n1}(o,da), \ldots, 1+\omega_{nn}(o,da)$ of $\vec{e}_n$. These linearly independent forms $\omega_1(o,da), \ldots, \omega_n(o,da), \omega_{11}(o,da), \ldots\ldots, \omega_{nn}(o,da)$ will be chosen as the components of the infinitesimal transformation S_{da}.

Let a repère R_a, composed of $[P, \vec{e}_1, \ldots, \vec{e}_n]$, and another repère R_{a+da}, composed of $[P+dP, \vec{e}_1+d\vec{e}_1, \ldots, \vec{e}_n+d\vec{e}_n]$ be transformed by the transformations S_a^{-1}. We shall obtain the repère R_0 and a repère R' which is composed of a point P' and n vectors $[\vec{e}_1{}', \ldots, \vec{e}_n{}']$. By definition,

$$\overrightarrow{OP}' = \omega_1(a,da)\vec{e}_1^{\,o} + \ldots\ldots + \omega_n(a,da)\vec{e}_n^{\,o},$$

$$\vec{e}_1{}' - \vec{e}_1^{\,o} = \omega_{11}(a,da)\vec{e}_1^{\,o} + \ldots\ldots + \omega_{1n}(a,da)\vec{e}_n^{\,o},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\vec{e}_n{}' - \vec{e}_n^{\,o} = \omega_{n1}(a,da)\vec{e}_1^{\,o} + \ldots\ldots + \omega_{nn}(a,da)\vec{e}_n^{\,o}.$$

According to the properties of the affine group, it follows that

$$\boxed{\begin{aligned} \overrightarrow{dP} &= \sum_{i}^{1,\ldots,n} \omega_i(a,da)\vec{e}_i \\ \overrightarrow{de_i} &= \sum_{j}^{1,\ldots,n} \omega_{ij}(a,da)\vec{e}_j \end{aligned}}$$

These relations determine the relative components of the '<u>repère mobile</u>' of an affine group if we know the relative positions of the two <u>repères</u> R_a and R_{a+da}.

<u>Equations of structure of E. Cartan.</u>

7. Consider a finite continuous group of transformations S_a. Let $\omega_1(a,da)$,............, $\omega_r(a,da)$ be the relative components of the infinitesimal displacement of its '<u>repère mobile</u>'.

We can imagine, in the representative variety of r dimensions of the transformations of the group G_r, two simply transitive groups, namely the groups of parameters

$S_\xi' = S_a S_\xi$ ———— First parameter group

and $S_\xi' = S_\xi S_a$ ———— Second parameter group

These groups of parameters leave the relative and absolute components ω_r and $\tilde{\omega}_r$ invariants, that is to say, we have

$\omega_\rho(\xi', d\xi') = \omega_\rho(\xi, d\xi)$ by the first parameter group

and $\tilde{\omega}_\rho(\xi', d\xi') = \tilde{\omega}_\rho(\xi, d\xi)$ by the second parameter group

Therefore

$$d\,\omega_\rho(\xi', \delta\xi') - \delta\,\omega_\rho(\xi', d\xi') = d\,\omega_\rho(\xi, \delta\xi) - \delta\,\omega_\rho(\xi, d\xi)$$

Now, each of these '<u>covariant bilinear</u>', is a '<u>alternate bilinear</u>' form with respect to the differentials of the independent variables, and is, therefore, also with respect to the forms ω_ρ.

Therefore, we have

$$\omega'_\rho(\xi, d\xi, \delta\xi) = \sum_{(p,q)} C_{pq\rho}(\xi)\left[\omega_p(\xi, d\xi)\,\omega_q(\xi, d\xi)\right],$$

where

$$C_{pq\rho}(\xi) + C_{qp\rho}(\xi) = 0.$$

Similarly

$$\omega'_\rho(\xi', d\xi', \delta\xi') = \sum_{(p,q)} C_{pq\rho}(\xi')\left[\omega_p(\xi', d\xi')\,\omega_q(\xi', d\xi')\right]$$

Therefore

$$C_{pq\rho}(\xi') = C_{pq\rho}(\xi).$$

We can choose any two arbitrary points for ξ and ξ' of the space of parameters. The quantities $C_{pq\rho}(\xi)$ are, therefore, independent of the variables $\xi_1, \ldots\ldots, \xi_r$.

These constants $C_{pq\rho} = -C_{qp\rho}$ are the constants of structure of the r-parameter group.

<u>Theorem of Structure</u>:-

Being given a group, the components of the infinitesimal displacements of its '<u>repère mobile</u>' are subjected to verify the equations of structure

$$\boxed{\omega'_\rho = \sum_{(p,q)} C_{pq\rho}\,[\omega_p \omega_q]}$$

The absolute components verify the relations.

$$\boxed{\tilde{\omega}'_\rho = -\sum_{(p,q)} C_{pq\rho}\,[\tilde{\omega}_p \tilde{\omega}_q]}$$

These equations of structure were established for the first time by Maurer. But the importance of these equations was realized by E. Cartan who pointed out that these equations reveal the structure of the relative as well as of the absolute components of the '<u>repère mobile</u>' of a group. For this reason these equations are called the equations of structure of E. Cartan.

<u>Equations of structure of the affine group of the space.</u>

8. The affine <u>repère</u> is composed of a point P and n vectors $[\vec{e}_1, \ldots, \vec{e}_n]$. The n(n+1) relative components of the displacements of the '<u>repère mobile</u>' are defined geometrically by the following formulae

$$\boxed{\begin{aligned} \vec{dP} &= \sum_K \omega_K \vec{e}_K \\ \vec{de}_i &= \sum_K \omega_{iK} \vec{e}_K \end{aligned}}$$

By writing the conditions of integrability of these equations

$$d\delta\vec{P} - \delta d\vec{P} = 0,$$
$$d\delta\vec{e}_i - \delta d\vec{e}_i = 0,$$

we get the following equations of structure of the affine group

$$\boxed{\begin{aligned} \omega_i' &= \sum_{K}^{1,\ldots,n} [\omega_K \omega_{Ki}] \\ \omega_{ij}' &= \sum_{K}^{1,\ldots,n} [\omega_{iK} \omega_{Kj}] \end{aligned}}$$

By taking the group of displacements of the space, we can easily see that these equations reduce to the equations of Darboux. Therefore, the equations of Darboux are the particular case of the equations of structure of Maurer - Cartan.

Fundamental identity.

9. The '<u>covariant bilinear</u>' of the first member of the equations of structure of E. Cartan

$$\omega_\rho' = \frac{1}{2}\sum_{p,q} C_{pq\rho}[\omega_p \omega_q] \quad (1 \leq \rho \leq r,\ 1 \leq p \leq r,\ 1 \leq q \leq r)$$

is zero, because it is itself a '<u>covariant bilinear</u>', of the forms of Pfaff. The '<u>covariant bilinear</u>' of the second member is

$$\frac{1}{2}\sum_{p,q} C_{pq\rho}[\omega_p' \omega_q] - \frac{1}{2}\sum_{p,q} C_{pq\rho}[\omega_q' \omega_p]$$
$$= \frac{1}{2}\sum_{p,q,\beta} C_{pq\rho} C_{\alpha\beta p}[\omega_\alpha \omega_\beta \omega_q].$$

we have, therefore, by replacing q by γ

$$\sum_{\alpha,\beta,\gamma,p} C_{\alpha\beta p} C_{p\gamma\rho}[\omega_\alpha \omega_\beta \omega_\gamma] = 0,$$

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On the geometry of the spaces of Riemann constructed by representing hyperquadrics as points.

Consider a space of n dimensions E with a fundamental group G of order r. If, according to a given law, we attach a repère to each point of the space having this point for its origin, the relative components ω_i of the infinitesimal displacement of this repère are the r forms of Pfaff which satisfy the equations of structure of G_r. These forms are constructed with the variables x_i (coordinates of a point) and their differentials. It is important to remark that if the sub-group g of the group G_r leaving the origin invariant be generated by the last r-n infinitesimal transformations of the group, the first n forms $\omega_1, \ldots, \omega_n$ are linearly independent.

We can adopt a different point of view:

Let us take a priori and expressions of Pfaff ω_i, constructed with n variables ($x_1, \ldots x_n$) and their differentials, satisfying the following conditions: ?

(1). they verify the equations of structure of G_r,

(2). the first n forms are linearly independent.

Let us take, in the space E, a repère R_x corresponding to a given system of values of x_i. The forms $\omega_i(x, dx)$ are the relative components of the infinitesimal transformation $S_{x;\ dx}$ of the group G_r. This transformation is defined analytically with respect to the repère R_x. Let R_{x+dx} be the repère corresponding to the system of values (x_i+dx_i). This repère is obtained from the repère R_x by the transformation $S_{x\ ;\ dx}$. The hypothesis that the ω_i satisfy the equations of structure assures that we always arrive to the same repère R_x from a given repère R_0 when we pass from the point, corresponding to the latter, to the point, corresponding to the former, by any continuous path whatsoever; the hypothesis that the forms $\omega_1, \omega_2, \ldots\ldots, \omega_n$, are linearly independent assures that the origin of the repères R_x fill up the whole space (or at least a region of the space).

Thus being given, in a continuum E_n, r forms of Pfaff $\omega_i(x,dx)$ satisfying the equations of structure of G_r, we say that the continuum possesses a structure, and the continuum is called a space based on the fundamental group G_r.

'Non-holonome' spaces.

11. Thus we arrive ~~to~~ at the notions of the generalized 'non-holonome' spaces of a given fundamental group G. We imagine that a space based on a fundamental group G is attached to each point A of a continuum. Imagine that there exists a law which permits to connect two such spaces attached to two infinitely near points of the continuum. Due to this law the portion of the continuum near the point A can be regarded as a portion of the space of Klein almost

to the second order. The space attached to the point A can be called the space tangent to the continuum at the point A. If we consider a continuous path, in the continuum, from a point A to another point B, the connection of the space tangent at B with the space tangent at A can be made differentially along the path under consideration and we say that this connection constitutes the development of the path AB and of the tangent spaces of the continuum situated in the immediate neighbourhood of the path AB on the tangent plane at A. If we take another path from A to B, then the development obtained along this new path will not coincide with the first in general. We can say that the given continuum, with a law permitting the connection of the tangent spaces, constitutes a '<u>non-holonome</u>' space of which the fundamental group is G.

The '<u>non-holomie</u>' is expressed by the fact that if we develop a cycle, drawn in the continuum starting from A and coming back to it, on the tangent space at A, the point A and its neighbourhood are found to undergo a certain displacement (or transformation of G) in order to arrive to its initial position. This displacement will be called displacement associated with the cycle.

Let

$X_1f, X_2f, \ldots\ldots\ldots, X_rf$

where

$$\boxed{(X_i X_j)f = \sum_K^{1,\ldots,r} C_{ijK} X_K f}$$

be the infinitesimal transformations of the fundamental group G_r. When we make the connection of an infinitely near tangent space E_A' with the tangent space E_A, the point of coordinates ($x_1, \ldots\ldots, x_n$) of E_A, referred to the <u>repère</u> R_A, coincides with the point of coordinates ($x_i + dx_i$) of E_A'.

We have therefore

$$dx_i + \sum_K^{1,\ldots,r} \omega_K X_K(x_i) = 0 \qquad (i = 1,\ldots\ldots, n)$$

where ω_K are the expressions of Pfaff linear with respect to the n quantities ($u_1, \ldots\ldots, u_n$) which localize the point A in the given continuum.

If the continuum be a space of Klein, we have

$$\boxed{\omega_i' = \sum_{(jK)} C_{jKi} [\omega_j \omega_K]} ;$$

If the continuum be <u>non-holonome</u>, we get

$$\boxed{\omega_i' = \sum_{(jK)} C_{jKi} [\omega_j \omega_K] + \Omega_i}$$

the Ω_i being the '<u>exterior</u>' quadratic forms in $du_1, \ldots\ldots, du_n$.

These quantities Ω_i define the <u>curvature</u> and <u>torsion</u> of the space.

<u>Variety of affine connection.</u>

12. It is important to see how the geometry of the variety of affine connexion is inserted in the general scheme mentioned in the preceding paragraph.

The tangent space E is, in this case, the affine space and the fundamental group is the group of affine transformations generated by the infinitesimal transformations

$$\frac{\partial f}{\partial x_i} \; ; \; x^i \frac{\partial f}{\partial x_j}$$

which are n (n+ 1) in number.

Since

$$\sum_{\rho}^{1,\dots,r} \omega_\rho X_\rho f = \omega^i \frac{\partial f}{\partial x_i} + \omega_i^j x^i \frac{\partial f}{\partial x_j} ,$$

the equations

$$dx_i + \sum_{K}^{1,\dots,r} \omega_K X_K(x_i) = 0$$

take the following form

$$dx^i + \omega^i + \omega_j^i x^j = 0$$

By expressing that these equations are completely integrable, we have

$$\boxed{\begin{aligned} (\omega^i)' &= \sum_K [\omega^K \omega_K^i] \\ (\omega_i^j)' &= \sum_K [\omega_i^K \omega_K^j] \end{aligned}}$$

These equations define the structure of the affine space.

<u>Alternative method</u>: These formulae of the structure of affine space can also be obtained in the following manner:

The equations defining the relative components of affine group are given by

$$\overrightarrow{dP} = \sum_{K}^{1,\dots,n} \omega^{K} \overrightarrow{e_K}$$

$$\overrightarrow{de_i} = \sum_{K}^{1,\dots,n} \omega_i^{K} \overrightarrow{e_K}$$

The ω^{i} and ω_i^{j} are linear with respect to the differentials du_i, where u_i denote the parameters on which the <u>repère</u> of the affine space depend. These $n(n+1)$ forms of Pfaff permit to '<u>repérer</u>' the system of reference at $P+dP$ with respect to the system of reference at P. We can say that they define an infinitesimal displacement permitting to pass from the latter to the former.

In the case of an affine space the displacement associated to a closed contour is zero.

Therefore

$$d\delta\overrightarrow{P} - \delta d\overrightarrow{P} = (\omega^{i})' - \sum_{K} [\omega^{K}\omega_K^{i}] = 0$$

$$d\delta\overrightarrow{e_i} - \delta d\overrightarrow{e_i} = (\omega_i^{j})' - \sum_{K} [\omega_i^{K}\omega_K^{j}] = 0 .$$

But, in the case of the space of affine connexion, the displacement associated to an infinitely small cycle is given by

$$\boxed{\begin{aligned} (\omega^{i})' - \sum_{K} [\omega^{K}\omega_K^{i}] &= \Omega^{i} = A^{i}_{jK} [\omega^{j}\omega^{K}] \\ (\omega_i^{j})' - \sum_{K} [\omega_i^{K}\omega_K^{j}] &= \Omega_i^{j} = A^{j}_{iK\ell} [\omega^{K}\omega^{\ell}] \end{aligned}}$$

The vector $\Omega^i \vec{e}_i$ defines the <u>torsion</u> of the space of affine connection. If the geometric sum of an infinitely small cycle is zero, this <u>torsion</u> will be zero.

The forms Ω_i^j define the <u>curvature</u> of the space of affine connexion.

Thus these formulae define the structure of the space of affine connexion.

<u>Varieties of metric connexion.</u>

In his memoir " Annales de L'École Norm. Supérieure. t.40.,1923" E. Cartan has given the structure of the most general metric varieties. These are the varieties of affine connexion for each point of which the tangent space is an euclidean space. The mutual '<u>repérage</u>' of two tangent euclidean spaces at two infinitely near points is made by means of the following expressions of Pfaff

$$\boxed{\omega^i, \omega, \omega_i^j = -\omega_j^i \ (i \neq j)}$$

where

(1). ω^i are the components of translation,

(2). ω is the component of '<u>homothétie</u>', the ratio of homothétie being ($1+\omega$), and

(3). ω_i^j are the components of rotations which bring two infinitely near '<u>repères</u>' into coincidence.

The equations of structure of the metric space are given by

$$\boxed{\begin{aligned} &(\omega^{i})' - [\omega^{i}\omega] - \sum_{K}[\omega^{K}\omega^{i}_{K}] = \Omega^{i} \\ &\omega' = \Omega \\ &(\omega^{j}_{i})' - \sum_{K}[\omega^{K}_{i}\omega^{j}_{K}] = \Omega^{j}_{i} \end{aligned}}$$

where in ω^{j}_{K} $j \neq K$

Therefore, with any elementary cycle drawn in this variety are associated:

(1). a translation $\Omega^{i}\vec{e}_{i}$

(2). a 'homothétie' of ratio ($1+\Omega$), and

(3). a rotation of components Ω^{j}_{i}.

They define the torsion, the curvature of 'homothétie' and the curvature of the metric variety.

If the curvature of '<u>homothétie</u>' is zero, the variety is called the variety of euclidean connexion in the terminology of E. Cartan. The variety for which torsion and the curvature of <u>homothétie</u> are zero is the classical space of Riemann.

The equations of structure of the Riemannean variety are given by

$$\boxed{\begin{aligned} &(\omega^{i})' = \sum_{K}[\omega^{K}\omega^{i}_{K}] \\ &(\omega^{j}_{i})' - \sum_{K}[\omega^{K}_{i}\omega^{j}_{K}] = \Omega^{j}_{i} = R^{j}_{iK\ell}[\omega^{K}\omega^{\ell}] \end{aligned}}$$

Given the components ($\omega_1, \dots, \omega_n$) of the infinitesimal translation of a <u>rectangular repère</u> mobile, the first equations of structure permit to calculate the $\frac{1}{2}n(n-1)$ components of rotations ω_i^j without any ambiguity. Therefore, the space <u>non-holonome</u> of Riemannian connexion is completely determined by means of the components of translations which verify the first set of the equations of structure. This space is, therefore, determined by means of the given quadratic form.

$$ds^2 = \sum_{i}^{1,\dots,n} (\omega^i)^2 .$$

The elementary rotation ω_j^i, determined by means of $ds^2 = \sum_i (\omega^i)^2$, provides the mutual orientation of two infinitely near <u>repères</u>. They decide, therefore, the parallelism of two infinitely near vectors; thus we arrive ~~to~~ the notion of parallelism introduced by T. Levi-Civita.

<u>Generalization of the identities of Bianchi.</u>

14. There exists a theorem, due to E. Cartan, true in all <u>non-holonome</u> spaces of which the identities of Bianchi are particular case in the geometry of Riemann space. E. Cartan calls it ' <u>the theorem of Conservation of Curvature and torsion</u>'.

This theorem is deduced by applying the "<u>fundamental identities</u>" to the equations of structure of the variety. That is to say, given the equations of structure of the variety in the form

$$\omega_\rho' = \sum_{(ij)} C_{ij\rho} [\omega^i \omega^j] + \Omega_\rho$$

the '<u>exterior derivate</u>' of the second member of these equations namely

$$\boxed{\Omega_\rho' = C_{kh\rho} \left\{ [\Omega_h \omega_k] - [\omega_h \Omega_k] \right\}}$$

give the conservation theorem.

This theorem can be interpreted geometrically: If we consider an element of volume of the variety and the surface which bounds it, the infinitesimal transformations associated to different elements of this surface have a sum equal to zero.

<u>Group of holonomie</u>.

15. E. Cartan has introduced a very fruitful notion of the group of <u>holonomie</u> in the <u>non-holonome</u> spaces.

Let E_A be a <u>holonome</u> space tangent at the point A of a given <u>non-holonome</u> variety. The fundamental group is supposed to be given in a definite analytic form, geometrically corresponding to a certain system of reference or repère. In a space E based on a fundamental group G, we shall call '<u>normal repère</u>' a <u>repère</u> which is deduced from the preceding by an arbitrary transformation of the group G. Being given two '<u>normal repères</u>' R and R', the formulae which permit to pass from the coordinates of a point referred to R' to the coordinates of the same point referred to R define analytically a transformation S of the group. We say that S is the displacement which carries R to R'.

Let a '<u>normal repère</u>' be chosen in the <u>holonome</u> space E_A tangent at A to a given variety. After the discription of a cycle whose origin and the extremity are at A, the <u>repère</u> R_A takes a new position R'_A in the tangent sapce E_A. Let T be the displacement (a transformation of G) which carries R_A to R'_A. The aggregate of the transformations T generate a group called the group of <u>holonomie</u> associated to the point A. Let us choose another repère $\bar{R}_A$ in the tangent space E_A. This <u>repère</u> would take a new position $\bar{\bar{R}}'_A$ after the development of the cycle on the tangent space E_A. Let S be the displacement which carries R to $\bar{R}_A$ and also R'_A to $\bar{R}'_A$. The displacement which carries $\bar{R}_A$ to $\bar{R}'_A$ and also is defined analytically by the transformation $S^{-1}\,T\,S$. Therefore, if we choose any other <u>repère</u> $\bar{R}_A$, in the space E_A instead of the first repère R_A, the group will be replaced by the group $S^{-1}\,g\,S$ which is '<u>homologue</u>' of g.

<u>Theorem of homogeniety:</u>

The groups associated to different points of a variety are sub-groups of the fundamental group G, and these sub-groups are '<u>Homologous</u>'.

Let us take a second point B of the <u>non-holonome</u> space, and let R_B be a ' <u>normal repère</u>' chosen in the space E_B. Let us connect the space E_B with the space E_A by an arbitrary but definite path ACB going from A to B. The <u>repère</u> R_B is thus placed in the space E_A. Let S be the displacement which carries R_A to R_B. Let BDB be a cycle whose origin is B and T be the transformation of G which is associated to it in order to carry R_B to R_B'.

Along the cycle ACBDBCA, in which the path chosen from A to B is described twice in the opposite sense, the <u>repère</u> R_A is first brought to R_B, then to R_B' and finally in a position R_A', which is placed with respect to R_B' in the same relation as R_A is placed with respect to R_B. Therefore, to any transformation T associated to the point B there corresponds a transformation STS^{-1} of the group associated to the point A. The reciprocal of this result can be demonstrated in the same way. Hence the groups of <u>holonomie</u> associated to A and B are '<u>homologous</u>'.

C H A P T E R II.

On the definition of S-operator.

Enunciation of the problem.

16. Let us take a group G operating in a space of affine connexion. Considering that the group G transforms the points, the coordinates and not the vectors and tensors of the space, I have shown in a paper " One parameter continuous group of deformations"* that the new components of the vectors and tensors are determined by a suitable operator S associated with the given Lie group and defined by

$$\boxed{S T^{i}_{j} = u^{r} T^{i}_{j|r} + u^{r}_{|j} T^{i}_{r} - u^{i}_{|r} T^{r}_{j} + u^{r} T^{i}_{m} \Lambda^{m}_{rj} - u^{r} T^{m}_{j} \Lambda^{i}_{rm}},$$

where the infinitesimal transformation of the group is given by

$$Xf = u^{r} \frac{\partial f}{\partial x_{r}}$$

The problem can be exposed in a more geometrical manner: Let a repère ($\vec{e}_1, \dots\dots, \vec{e}_n$) be attached at every point x of the space E. Let us consider $\vec{V}$ and $\vec{V}'$ to be the vectors of a given vector field at the

* To be appeared shortly in Jour. Ind. Math. Soc.

points x and x' respectively. Let $(\vec{e}_1, \ldots\ldots, \vec{e}_n)$ be transformed into $(\vec{e}'_1, \ldots\ldots, \vec{e}'_n)$ at x' by the group G. If $\overset{*}{\vec{V}}$ be of components $(\overset{*}{V}{}^1, \ldots\ldots, \overset{*}{V}{}^n)$ with respect to the repère $(\vec{e}'_1, \ldots\ldots, \vec{e}'_n)$, it follows from the first principle that $V^{i*} = V^i$ $(i = 1, \ldots\ldots, n)$. Hence, referring the vectors to the <u>repère</u> $(\vec{e}'_1, \ldots \vec{e}'_n)$ at x', and calling the components of $\vec{V}'$, with respect to this <u>repère</u>, $\overline{V}^i$ we have

$$\boxed{\overline{V}^i = e^{\varepsilon S} V^i}$$

using the notation of H. Poincaré.

That is to say that the vectors and the tensors of the space E are transformed by a group whose infinitesimal transformation is defined by the S-operator.

It would not be without interest to mention that if we adopt a second point of view namely : G transforms in the space E <u>figures into figures</u>, <u>vectors into vectors</u>, <u>the repère remaining invariant</u>, then each vector of compoments V^i is transformed into a vector of components $\overline{V}^i$ where

$$\boxed{\overline{V}^i = e^{-\varepsilon S} V^i}$$

<u>Derivation of S-operator for contravariant vectors.</u>

17. According to the theory of E. Cartan we have

$$\overrightarrow{dP} = \sum_i \omega^i(d)\vec{e}_i$$

$$\overrightarrow{de_i} = \sum_j \omega_i^j(d)\vec{e}_j$$

defining small variations of the point P and the vectors ($\vec{e}_1, \ldots, \vec{e}_n$) of the <u>repère</u> at P, referred to the same <u>repère</u>.

Also the equations of structure of the space of affine connexion are given by

$$(\omega^i)' - \sum_K [\omega^K \omega^i_K] = \Omega^i = \sum_{(jK)} \Lambda^i_{jK} [\omega^j \omega^K]$$

$$(\omega^j_i)' - \sum_K [\omega^K_i \omega^j_K] = \Omega^j_i = \sum_{(K\ell)} \Lambda^j_{iK\ell} [\omega^K \omega^\ell].$$

Taking

$$dP = \vec{V}(V^1, \ldots, V^n) \ , \ \delta P = \varepsilon U(u^1, \ldots, u^n)$$

and using the first equations of structure we have

$$\Delta_u V^i - D_V u^i = \Lambda^i_{jK} u^j V^K$$

But

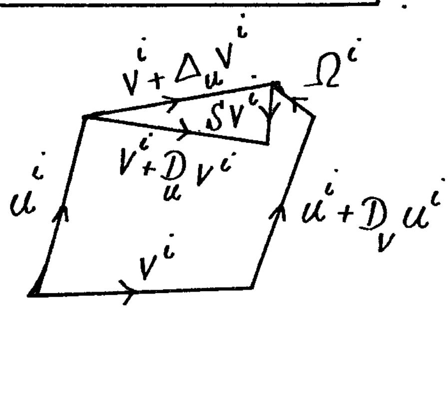
$$S V^i = D_u V^i - \Delta_u V^i$$

i.e.,

$$S V^i = D_u V^i - D_V u^i - \Lambda^i_{jK} u^j V^K$$

or,

$$\boxed{S V^i = u^\gamma V^i_{|\gamma} - u^i_{|\gamma} V^\gamma - \Lambda^i_{jK} u^j V^K}$$



Derivation of S-operator for mixed tensors.

18.

Since

$$\mathcal{S}(W_i V^i) = \mathcal{S}(W_i)V^i + \mathcal{S}(V^i)\, W_i = u^\gamma \frac{\partial}{\partial x_\gamma}(W_i V^i),$$

Therefore

$$\boxed{\mathcal{S} W_i = u^\gamma W_{i|\gamma} + u^\gamma_{|i} W_\gamma + u^\gamma W_m \Lambda^m_{\gamma i}}$$

The operator S having been thus defined for covariant and contravariant vectors, we can easily extend its definition to any mixed tensor T^i_j as given in paragraph 16.

Variation of the expressions $\omega^j_i(d)$.

19.

Using the second equations of structure of the space E and taking

$$\overrightarrow{d e_i} = \gamma^h_{ik}\, \omega^k(d)\, \vec{e}_h$$

$$\overrightarrow{\delta e_i} = \varepsilon\left(u^h_{|i} + u^\gamma \Lambda^h_{\gamma i}\right)\vec{e}_h$$

we get

$$\delta\omega^h_i(d) = \left(\bar{\gamma}^h_{ik} - \gamma^h_{ik}\right)\omega^k(d) = \varepsilon\, a^h_{ik}\, \omega^k(d),$$

where

$$\boxed{a^h_{ik} = u^h_{|i|k} + u^\gamma \Lambda^h_{i\gamma k} + \left(u^\gamma \Lambda^h_{\gamma i}\right)_{|k}}$$

Thus

$$\boxed{\delta\gamma^h_{ik} = \varepsilon\, a^h_{ik} + \frac{\varepsilon^2}{2!}\,\mathcal{S} a^h_{ik} + \cdots}$$

S E C O N D   T H E S I S

On the study of the Lorentz-invariant spaces.

I N T R O D U C T I O N.

The theory of the spaces of Riemann has been made the object of various important works due to Christoffel, Lipschitz, Schur and Voss during the later part of the 19th century. Ricci and Levi-Civita introduced a powerful systematic method of "Absolute differential calculus" and developped the theory of the spaces of Riemann by their algebraic and analytic processes. The chief aim of this branch of mathematics, commonly called "Tensor analysis", is to construct and discuss such laws which are generally covariant. The method of Ricci and Levi-Civita became the object of a very widespread interest after its utilization in mathematical physics and in the theory of generalized relativity by A.Einstein (1916).

The fundamental memoir of Levi-Civita "Nozione di parallelismo in una varieta qualunque, Rend. Circ. matem. Palermo, t.42, 1917, p.173-205" gave rise to the generalization of the spaces of Riemann by its fruitful notion of differential parallelism. Levi-Civita imagined the metrical space of Riemann as hypersurface embedded in a euclidean space of higher dimensions. P(x^i) and P$'$($x^i + dx^i$) being the points of Riemann space and $\overrightarrow{PQ}$ a vector, Levi-Civita transfers $\overrightarrow{PQ}$ by a euclidean parallel shift to $\overrightarrow{P'Q_E}$

and defines $\overrightarrow{PQ''}$, the orthogonal projection of $\overrightarrow{PQ'}_E$ upon the euclidean plane tangential to the Riemann space at P, as parallel to $\overrightarrow{PQ}$. In spite of this artificial definition, parallelism is intrinsic and expressible by the metric of the Riemann space alone. Thus if λ^i and $\lambda^i + \delta\lambda^i$ be the components of $\overrightarrow{PQ}$ and $\overrightarrow{PQ''}$,

$$\boxed{\delta\lambda^i + \Gamma^i_{jk}\,\lambda^j dx^k = 0}$$

For covariant components the law of parallelism is defined by

$$\boxed{\delta\lambda_i - \Gamma^k_{il}\,\lambda_k dx^l = 0}$$

These are the formulae giving the laws of parallel transport of the contravariant and covariant vectors in the Riemannian space regardless of the coordinate system used.

A direct explanation of this conception was given by H. Weyl with the help of the geodesic coordinate system where the metric tensor g_{ij} is stationary at the point taken as the origin of the vector λ^i; the laws of parallelism being then represented by the formulae:

$$\boxed{\delta\lambda^i = 0 \quad ; \quad \delta\lambda_i = 0}$$

In the paper " Reine infinitesimalgeometrie" H. Weyl elevated the idea of parallel displacement of a vector as introduced by Levi-Civita to the rank of a fundamental axiom which the former mathematician called the characteristic of the affine geometry.

According to H. Weyl parallelism may be defined in the non-metrical space by putting

$$\boxed{\delta\lambda^{i} + \Gamma^{i}_{k\ell}\lambda^{k}dx^{\ell} = 0}\,,$$

where $\Gamma^{i}_{k\ell} = \Gamma^{i}_{\ell k}$ are the free prescribable functions of position; fixing the affine connection of the space. The equations giving the transformation of the coefficients of connection ($\Gamma^{i}_{k\ell}$) from one coordinate system to the other are the same as those for the three-index symbols of Christoffel in Riemannian geometry. Thus the covariant differentiation is generalized for the spaces of affine connection.

One can impose metrics upon the affine manifold. Then the requirement that the size of a vector remain unaltered by a translation :

$$\delta(g_{ij}\lambda^{i}\lambda^{j}) = 0,$$

gives $\Gamma^{i}_{k\ell} = \{k\ell, i\}$ and leads to the differential law of parallelism in Riemann spaces

$$\boxed{\delta\lambda^{i} + \{k\ell, i\}\lambda^{k}dx^{\ell} = 0}\,.$$

The researches of J.A. Schouten and E. Bompiani show that the concept of differential parallelism throws a light on the notion of the space curvature, considered as showing the non-integrability of the transport by parallelism.

Veblen and Eisenhart in their papers dealing with the geometry of paths have given a new generalization of the geometry of Riemannian spaces. One can uniquely determine a path; considered as the straight lines of an affinely connected space, by a point and a direction or by two infinitely near points. Conversely, we say that a system of curves possessing the above mentioned property can be considered as the straight lines of a space and we can deduce an affine connection of the space therefrom.

Various other generalizations of the geometry of paths have been given by Berwald and Douglas by taking for the paths the integral curves of a system of differential equations. It is not ~~of~~ these new geometries that I shall discuss here but I shall briefly mention the methods of Kosambi for the geometry of the generalized path-spaces and show how his results are in agreement with the results of E. Cartan deduced by his own methods based on the theory of Pfaff's form. This is due to the fact that there is no difference between the foundation of both the methods. Both methods are based on the alternation of the fundamental operations.

Even then we can say that the method of Kosambi is a bit more illuminating for it sets up a basis for the tensorial operators and by alternating these settles completely the question of the fundamental set of differential invariants which are independent of each other whereas the method of Cartan does not throw any light on the question of independence or interdependence of the differential invariants of the space.

In Chapter I I shall give (i) the two main axioms which Kosambi has utilized in order to develope the intrinsic differential geometry of the path-spaces (ii) the fundamental tensorial operators and (iii) the set of independent differential invariants of the space. I shall also point out how the question of metric for the path-spaces has been settled by Kosambi. In Chapter II I shall show how these differential invariants of the generalized path-spaces can be obtained by the methods of E. Cartan mainly due to his own researches in the theory of continuous groups.

In the theory to which I have devoted this part of the work, the Kosambian new theory of relativity plays a fundamental part. It seems advisable, therefore, to give a brief account of this important subject in Chapter III; I shall not, however, attempt to do more than to give a clear statement of those ideas and theorems which Kosambi has given in his two memoirs " Path-equations admitting the Lorentz group I, II"; the first memoir appeared in the Jour.Lond.Math.Soc.15 (1940), pp. 86-91 and the second in the Jour.Ind.Math.Soc. Vol.V, No.2 (1941), pp. 62-72.

In the paper " On the existence of a metric for path-spaces admitting the Lorentz-group, Proc.Ind. Acad. Sc.Vol.XIII (1941) pp. 203-210" which makes Chapter IV of the work here, I have taken Kosambi's form of path-equations admitting the Lorentz-group:

$$\ddot{x}^i - p^i H(X,Y,Z) + \dot{x}^i J(X,Y,Z) = 0$$

where H, J are arbitrary, and applied to them the general condition that a set of paths be the extremals of a variation problem, namely, that their equations of variation be self-adjoint. A study of the resulting differential equations gives rise to a set of necessary and sufficient conditions for the existence of a metric and a method of solution. I have, then, applied the same method of attack to more specialized paths as those with the most general Lorentz invariant symmetric affine connection where

$$H = \frac{aY}{X} + \frac{bZ^2}{X^2}, \quad J = \frac{2CZ}{X}$$

This leads to simpler conditions which are in conformity with the results of Kosambi obtained by him by a direct method in his memoir already cited "Path-equations admitting the Lorentz-group II."

In my paper "Einstein spaces admitting the Lorentz-group, Proc.Ind.Acad.Sc., Sec.A, 14, pp. 133-138 (1941)" I start from an explicit form, involving two arbitrary functions α, γ of the most general Riemannian metric admitting the n-dimensional Lorentz-group. I have shown, then, that for $n \geqslant 4$ the Lorentz-invariant Riemann spaces are conformally flat and consequently this result indicates that a transformation of coordinates exists such that the metric can be transformed into the canonical form. The conditions that this space be an Einstein space reduce to one differential equation in α and γ which implies projective flatness and has an explicit solution when we impose Milne's condition on the paths. Finally I have taken the relativity field equations with stress energy tensor and have obtained formulae for the pressure and density in terms of α and γ.

In Chapter VI " Equations of Killing for Lorentz-invariant Riemann spaces, Jour.Osm.University Vol. X(1942)" I have classified the metric spaces according to the groups they admit. The vector generators of the groups of motions for each class of spaces have been explicitly obtained and the structures of the various groups arising therein have been studied. The same method has, afterwards, been applied to the Lorentz-invariant path-spaces with a symmetric affine connection in Chapter VII " On the classification of the Lorentz-invariant spaces according to the groups they admit, Jour.Osm. University, Vol.X, (1942)" and the spaces have been classified into the metric and non-metric spaces.

In 1925, Eisenhart in his epoch making paper, which appeared in Nat.Acad.Sci.Proc.Vol.II pp. 246-250, showed that the linear connections of space can be determined by simply transitive groups. After the appearance of this paper Cartan and Schouten studied the geometry of the group-space in view of the fact that the parameter groups of a group are simply transitive and reciprocal, and hence the methods of Eisenhart could be applied to define two asymmetric connections and a symmetric one for the group-space. Eisenhart defines functions L^{i}_{jk} by means of the components u^{i}_{a} of n vectors of a simply transitive group G_n in a space V_n in the following manner :

$$\boxed{L^{i}_{jk} = u^{i}_{a}\frac{\partial u^{a}_{k}}{\partial x^{j}} = -u^{a}_{k}\frac{\partial u^{i}_{a}}{\partial x^{j}} \qquad (a,i,j,k = 1,\ldots,n)}$$

which transform under a non-singular transformation of coordinates like the coefficients of asymmetric connection. The curvature tensor of the linear connection, thus determined, is identically zero:

$$L^{i}_{jkl} \equiv 0$$

By making use of these results of Eisenhart I have obtained a generalization of a theorem of Bianchi, viz., " Any simply transitive group in n variables is the group of motions (complete or partial) of an infinity of Riemann spaces V_n" for the Kosambian path-spaces in Chapter VIII on " Simply transitive groups as groups of collineations in Path-spaces, Jour.Osm.University Vol.X, 1942".

C H A P T E R I.

Generalized Path-spaces and Methods of Kosambi.

(1). In his memoir "Parallelism and Path-spaces, Math. Zeitschrift, Bd.37(1933), pp.608-618" Kosambi has shown that an intrinsic differential geometry can be associated with an arbitrary system of second order differential equations of the form:

$$\text{(I)} \qquad \ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0, \quad i = 1, \dots, n$$

$$\dot{x}^i = \frac{dx^i}{dt}$$

The solution curves of (I) are regarded as "paths" of n-dimensional space (x) and possess the property that one and only one such curve is determined by a point and a direction, or two points, within some sufficiently restricted n-dimensional region of the space.

(2). Two main assumptions are needed to develop the intrinsic differential geometry of the space. These are:

1. the tensor invariance of all fundamental equations including (I) under the transformation group

$$\bar{x}^i = \bar{x}^i(x), \quad \bar{t} = t \text{ and}$$

2. the existence of a vectorial operator D, the derivate Du^i whose vanishing is to determine the parallel displacement of u^i, along a path. We require of this parallelism, the following postulates:

(i) Linearity in the derivate

$$\mathcal{D}u^i = \dot{u}^i + \beta^i(x, \dot{x}, u, t),$$

(ii) the vectorial law of transformation

$$\bar{\mathcal{D}}\bar{u}^i = F^i_{\gamma}\,\mathcal{D}u^{\gamma}$$

where $F^i_{\gamma} = \dfrac{\partial \bar{x}^i}{\partial x^{\gamma}}$ and $F \equiv \|F^i_{\gamma}\| \neq 0,$

(iii) auto-parallelism for the paths

$$\mathcal{D}\dot{x}^i \equiv \ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0.$$

Expanding the second of these it follows that

$$\dot{\bar{u}}^i + \beta^i(\bar{x}, \dot{\bar{x}}, F^i_{\gamma} u^{\gamma}, t) \equiv F^i_{\gamma}\,(\dot{u}^{\gamma} + \beta^{\gamma}).$$

The functional equation is an identity in u and u, ?
and can be solved by inspection with the following
results:

$$\beta^i = u^k \gamma^i_k(x, \dot{x}, t) + \varepsilon^i(x, \dot{x}, t)$$

so that

$$\mathcal{D}u^i = \dot{u}^i + u^k \gamma^i_k(x, \dot{x}, t) + \varepsilon^i(x, \dot{x}, t).$$

ε^i is a vector so that

$$\bar{\varepsilon}^i = F^i_j\,\varepsilon^j$$

Also

$$F^k_j\,\bar{\gamma}^i_k + \frac{d}{dt} F^i_j = \gamma^k_j F^i_k.$$

Since the vanishing of the derivate of x gives the paths, therefore

$$\mathcal{D}x^i = \ddot{x}^i + \dot{x}^k \gamma^i_k + \mathcal{E}^i \equiv \ddot{x}^i + \alpha^i = 0$$

so that

$$\mathcal{E}^i = \bar{\alpha}^i - \gamma^i_k \dot{x}^k$$

(3). The omission of the vector $\mathcal{E}^i$ still gives a vectorial operator, one that can be defined by analogy with ordinary covariant differentiation for covariant vectors, and hence for general tensors

$$\mathscr{D}u^i \equiv \dot{u}^i + \gamma^i_k u^k,$$

$$\mathscr{D}u_i \equiv \dot{u}_i - \gamma^k_i u_k$$

The operator $\mathscr{D}$, called the bi-derivate, is distributive whereas the original one is not.

(4). The determination of the intrinsic differential invariants of the space requires the determination of the connection γ^i_k more closely. To this end, we demand that the equations of variation of (I) be reducible to as simple a form as possible. This is justified in as much as these equations; which give the geodesic deviation are our only means of exploring the space in the neighbourhood of any given path. The equations of variation of (I)

(II) $\quad \ddot{u}^i + \alpha^i_{;r}\dot{u}^r + \alpha^i_{,r}u^r = 0$

can be put in the form

$$\mathcal{D}^2u^i = P^i_r u^r - 2\sigma_r{}^i\mathcal{D}u^r + u^r\mathcal{D}\sigma_r{}^i + \sigma_\kappa{}^i\sigma_r{}^\kappa u^r.$$

Here

$$\sigma_r{}^i = \tfrac{1}{2}\alpha^i_{;r} - \gamma^i_r$$

$$P^i_j = -\alpha^i_{,j} + \tfrac{1}{2}\alpha^i_{,\kappa;j}\dot{x}^\kappa + \tfrac{1}{2}\frac{\partial}{\partial t}\alpha^i_{;j} - \tfrac{1}{2}\alpha^i_{;j;\kappa}\alpha^\kappa + \tfrac{1}{4}\alpha^i_{;r}\alpha^r_{;j}$$

Both of these are seen to be tensors. The first is an invariant of the connection, the second an invariant of the space, or an intrinsic invariant. Taking

$$\gamma^i_j = \tfrac{1}{2}\alpha^i_{;j}$$

we have our desired simplest form

$$\mathcal{D}u^i \equiv \dot{u}^i + \tfrac{1}{2}\alpha^i_{;r}$$

$$\mathcal{D}^2u^i = P^i_r u^r$$

The mixed tensor P^i_j corresponds to the Riemann-Christoffel tensor in this scheme.

(5). The fundamental tensorial operations for the path-space are:

$$\frac{\partial}{\partial \dot{x}}\,;\ \mathcal{D}\,;\ \frac{\partial}{\partial t}$$

We find on alternating upon a vector u^i

$$u^i_{;j;\kappa} - u^i_{;\kappa;j} = 0$$

$$\frac{\partial}{\partial t}u^i_{;j} - \left(\frac{\partial u^i}{\partial t}\right)_{;j} = 0\ .$$

By alternating $\mathcal{D}$ and $\frac{\partial}{\partial \dot{x}}$ we get a differential operator playing the role of covariant differentiator

$$(\mathcal{D}u^i)_{;k} - \mathcal{D}(u^i_{;k}) = u^i_{|k}$$

where

$$u^i_{|k} = u^i_{,k} - \tfrac{1}{2} u^i_{;r}\alpha^r_{;k} + \tfrac{1}{2} u^r \alpha^i_{;r;k}$$

By alternating the fundamental tensorial operators we get the following list of the fundamental intrinsic invariants of the space :

$$\dot{x}^i \; ; \; P^i_j \; ; \; \tfrac{1}{2}\alpha^i_{;j;k;l}$$

The other invariants can be expressed in terms of these fundamental invariants. For example we have

$$\varepsilon^i = -\mathcal{D}\dot{x}^i = \alpha^i - \tfrac{1}{2}\alpha^i_{;r}\dot{x}^r$$

$$R^i_{jk} = \tfrac{1}{3}\left(P^i_{j;k} - P^i_{k;j}\right)$$

$$\frac{\partial \alpha^i}{\partial t} = \mathcal{D}\varepsilon^i + P^i_j \dot{x}^j .$$

Metric for the path-spaces.

(6). A metric for our path-spaces defined by the integral curves of

(I) $$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0$$

will be taken to be the integrand of some variational principle, the extremals being the equations of Paths (I).

It has been shown by D.R.Davis " Trans. Am. Math. Soc., Vol. 33, p. 246 and Bull. Am. Math. Soc. 1929, pp. 371-380" that a necessary and sufficient condition for the existence of such a principle is the self-adjoincy of the equations of variation of (1), viz.,

$$\text{(II)} \qquad \ddot{u}^i + \alpha^i_{,r}\dot{u}^r + \alpha^i_{,r}u^r = 0$$

which are obtained when one varies the paths into nearby paths by means of the transformations

$$\bar{x}^i = x^i + u^i \delta\tau$$

Following the Lagrangian procedure of finding the adjoint system by finding out the integrating factor of the equations (II), we obtain that the equations

$$\ddot{v}_i - \frac{d}{dt}\left(v_r \alpha^r_{;i}\right) + v_r \alpha^r_{;i} = 0$$

or, $$\mathscr{D}^2 v_i = P^r_i v_r$$

are the adjoint systems of the equations of variation of the paths.

Because u^i is contravariant vector whereas v_i is covariant, it follows that the self-adjoincy evidently conveys the meaning of the process of association in our path-spaces.

In his paper " Systems of Differential equations of the second order. Quart. Jour. Maths. Oxford. 6. (1935) pp. 1-12" Kosambi has shown that any plausible formulation of such a law for raising and lowering indices inevitably leads to simple conditions for the existence of a metric. He obtains a set of necessary and sufficient conditions for the existence of a metric for the path-spaces

$$f_{;i;r} P^r_j - f_{;r;j} P^r_i = 0,$$

$$\mathcal{D} f_{;i;j} = 0,$$

$$| f_{,i;j} | \neq 0,$$

where the function f is taken to be the integrand of any regular problem of the calculus of variations whose extremals are defined by the equations (I).

C H A P T E R II.

Differential invariants and the theory of Pfaff's form.

(7). I consider, now, Cartan's method of generating the differential invariants of the generalized path-spaces. We consider the space of 2n+ 1 dimensions in $(x, \dot{x}, t)$, where t is considered to be an absolute time-like parameter.

E. Cartan considers the Pfaffians:

$$\omega^i(d) = dx^i - \dot{x}^i dt,$$

$$\theta^i(d) = d\dot{x}^i + \alpha^i dt + \gamma^i_r \omega^r(d),$$

$$\omega^i_j(d) = \gamma^i_j dt + \gamma^i_{jk}\omega^k(d),$$

the last of these defining the connection of the space.

By making use of the equations of structure of the space

$$\Omega^i = (\omega^i)' - [\omega^k \omega^i_k],$$

$$\Theta^i = (\theta^i)' - [\theta^k \omega^i_k],$$

$$\Omega^i_j = (\omega^i_j)' - [\omega^k_j \omega^i_k],$$

we get all the differential invariants of our path-spaces.

We find that

$$\Omega^i = -[\theta^i(\delta)dt - \theta^i(d)\delta t] - (\gamma^i_{kj} - \gamma^i_{jk})\omega^k(\delta)\omega^j(d).$$

If we take

$$\gamma^i_{kj} = \gamma^i_{jk}$$

we have

$$\Omega^i = -[\theta^i(\delta)dt - \theta^i(d)\delta t].$$

E. Cartan makes this restriction on the functions in order to get symmetric Christoffel symbols.

Then we get

$$\begin{aligned}\Theta^i = {} & [\theta^k(\delta)dt - \theta^k(d)\delta t](\alpha^i_{;k} - 2\gamma^i_k) \\ & + [\omega^h(\delta)\theta^k(d) - \omega^h(d)\theta^k(\delta)](\gamma^i_{kh} - \gamma^i_{h;k}) \\ & + (\omega^k(d)\delta t - \omega^k(\delta)dt)A^i_k \\ & + [\omega^k(d)\omega^h(\delta) - \omega^k(\delta)\omega^h(d)]A^i_{kh},\end{aligned}$$

where

$$A^i_k = \gamma^r_k\alpha^i_{;r} - \alpha^i_{,k} - \gamma^i_{k;r}\dot{\alpha}^r + \gamma^i_{k,r}\dot{x}^r + \frac{\partial}{\partial t}\gamma^i_k - \gamma^i_r\gamma^r_k$$

$$A^i_{kh} = \gamma^i_{k,h} - \gamma^r_h\gamma^i_{k;r}$$

E. Cartan makes the cofficients of the first two terms vanish in order to determine the intrinsic connection of the space.

Thus

$$\gamma^i_k = \frac{1}{2}\alpha^i_{;k}$$

$$\gamma^i_{hk} = \frac{1}{2}\alpha^i_{;h;k}$$

The other coefficients A^i_k and A^i_{kh}, then, become the mixed tensors

$$P^i_k \text{ and } R^i_{kh} = 1/3\,(P^i_{k;h} - P^i_{h;k})$$

of Kosambi.

Finally

$$\Omega^i_j = (P^i_{r;j} + R^i_{rj})[\omega^r(d)\delta t - \omega^r(\delta)dt] + R^i_{rh;j}\,\omega^r(\delta)\omega^h(d)$$
$$+ \tfrac{1}{2}\alpha^i_{;h;r;j}\,[\omega^h(d)\theta^r(\delta) - \omega^h(\delta)\theta^r(d)].$$

Thus we see that even by the methods of E. Cartan we obtain the same basic set of tensors, viz.,

$$\dot{x}^i \; ; \; P^i_j \; ; \; \tfrac{1}{2}\alpha^i_{;j;k;l}$$

CHAPTER III.

Lorentz-invariant path-spaces and various theories of Relativity.

(8). In his paper " Path-equations admitting the Lorentz group, Jour. Lond. Math. Soc. 15, 1940 pp. 86-91" Kosambi has shown that various theories of relativity, including that of Milne, can be obtained from a single unified point of view by regarding the universe and its world-lines defined by a general second order path-space of four dimensions

$$\ddot{x}^i + \alpha^i(x, \dot{x}) = 0, \; i = 0, 1, 2, 3$$

and by studying these paths directly with reference to the Lorentz group, the similitude group and the translation group.

The requirement of invariance under the rotation group whose generators are given by

$$x^j\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^j}$$

and whose first extension, and the extension relative to the paths, are represented by the operators

$$x^j\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^j}+\dot{x}^j\frac{\partial}{\partial \dot{x}^k}-\dot{x}^k\frac{\partial}{\partial \dot{x}^j}\ ;$$

$$x^j\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^j}+\dot{x}^j\frac{\partial}{\partial \dot{x}^k}-\dot{x}^k\frac{\partial}{\partial \dot{x}^j}-\alpha^j\frac{\partial}{\partial \ddot{x}^k}+\alpha^k\frac{\partial}{\partial \ddot{x}^j}$$

leads to the general form of the equations of Lorentz-invariant paths :

$$\ddot{x}^i-\dot{p}^i H(X,Y,Z)+\dot{x}^i J(X,Y,Z)=0$$

where H and J are arbitrary functions, and X,Y,Z are the absolute invariants of the group with respect to the first extension.

Invariance under Lorentz group plus similitudes gives the following form of the equations

$$\ddot{x}^i-\dot{p}^i H\left(\frac{Y}{X},\frac{Z}{X}\right)+\dot{x}^i J\left(\frac{Y}{X},\frac{Z}{X}\right)=0$$

which correspond to Milne's expanding universe.

If one considers the invariance under the Lorentz group and translations, one gets

$$\ddot{x}^i + \dot{x}^i H(Y) = 0$$

Considering the invariance under all the three groups, we obtain

$$\ddot{x}^i + c\dot{x}^i = 0,$$ c being constant,

as the equations of the trajectories. This case corresponds to Einstein's - Galilean space of special theory of relativity when c = 0.

These corresponding forms of the equations has, then, been derived by Kosambi for the case in which the parameter on the paths is the time coordinate, in order to include world structure deducible from three dimensional observations only. These equations, having been thus deduced from three dimensional observations, take the forms:

(1) $$\ddot{x}^i + p^i Y G\left(X, \frac{Z^2}{Y}\right) + \dot{x}^i Z \gamma\left(X, \frac{Z^2}{Y}\right) = 0,$$ Invariant under the rotation group.

(2) $$\ddot{x}^i + p^i \frac{Y}{X} G(\xi) + 2\dot{x}^i \frac{Z}{X} \gamma(\xi) = 0$$

where $\xi = \frac{Z^2}{XY}$, Invariant under the Lorentz group and the similitude

(3) $\ddot{x}^i + c\dot{x}^i\sqrt{y} = 0,$ Invariant under the Lorentz group and the translation.

In the last equations (3) c = 0 if a metric exists.

The spaces (2) are essentially those obtained by Milne, except that γ is arbitrary. If the world-lines of the fundamental particles are $\dot{x}^i = cx^i$ we have

$$1 - G(1) + 2\gamma(1) = 0 .$$

The arbitrary function γ can be determined by requiring the existence of a metric for these spaces.

Kosambi has obtained that the most general metric for Milne's case can be expressed in the form $\phi^2_{\xi}\frac{Y}{X}$, where $\phi(\log X, \xi)$ is a solution of the partial differential equation

$$\phi_{12} + (1+G)(1-\xi)\phi_{22} + (1+G)\left(\frac{3-4\xi}{2\xi}\right)\phi_2 = 0,$$

$$\phi_1 = \frac{\partial\phi}{\partial \log X}, \quad \phi_2 = \frac{\partial\phi}{\partial\xi} .$$

He has shown the validity of this formula even when $G = G(X, \xi)$ and the metrics found by Walker as well as arising when $\gamma = G = -1$ are included in his form of the metric.

CHAPTER IV.

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ON THE EXISTENCE OF A METRIC FOR PATH-SPACES ADMITTING THE LORENTZ GROUP*

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I. This note discusses the existence of a metric for the most general path-spaces admitting only the Lorentz group of the special theory of relativity.

I shall use the notation of Kosambi[1] throughout this note with the exception that X, Y, Z are used in place of the corresponding curly (Gothic) letters in his paper.

The equations, under discussion, have already been given by Kosambi[1] in the following form :—

$$\ddot{x}^i - p^i \, \mathrm{H}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) + \dot{x}^i \, \mathrm{J}(\mathrm{X}, \mathrm{Y}, \mathrm{Z}) = 0, \tag{1}$$

where H, J are arbitrary.

By a metric for the space defined by (1), we shall mean the integrand $f(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ of any regular problem of the calculus of variations whose extremals are defined by the equations (1). For the existence of such a metric, a set of necessary and sufficient conditions are derived from the condition that the equations of variation of (1) be self-adjoint.[2]

These conditions are :

$$\begin{aligned} &(a) \quad J_{;\,i;\,r} P^r_{\ j} - f_{;\,r;\,j} P^r_{\ i} = 0, \\ &(b) \quad \mathscr{D} f_{;\,i;\,j} = 0, \\ &(c) \quad |f_{;\,i;\,j}| \neq 0, \text{ where} \end{aligned} \tag{2}$$

$\mathscr{D}$ is the Kosambian derivate, $P^i_{\ j}$ is his mixed curvature tensor of the path-space and semicolon denotes differentiation with respect to $\dot{x}$.

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* I express my thanks to Prof. D. D. Kosambi for his kind guidance and helpful suggestions throughout the course of this investigation.

[1] D. D. Kosambi, " Path-equations admitting the Lorentz group," *Lond. Math. Soc.*, 1940, **15**, 86–91.

[2] D. D. Kosambi, *Quarterly Jour. Maths.*, 1935, **6**, 1–12 (See also the paper of D. R. Davis referred to therein).

The mixed tensor P^i_j for the space defined by (1) takes the following form :

$$P^i_j = A\delta_j^i + B\dot{x}^i \dot{x}_j + C\dot{x}^i p_j + D\dot{x}_j p^i + Ep^i p_j \; ; \qquad (3)$$

where,

$$A \equiv ZHJ_2 + \tfrac{1}{2} X HJ_3 - YJJ_2 - \tfrac{1}{2} ZJJ_3 + ZJ_1 + \tfrac{1}{2} YJ_3 + H + \tfrac{1}{4} J^2,$$

$$B \equiv 2 ZHJ_{22} + X HJ_{23} - JJ_2 - 2 YJJ_{22} - ZJJ_{23} - H_2 + 2ZJ_{12}$$
$$+ YJ_{23} - \tfrac{1}{2} J_3 - ZJ_2H_2 + YJ^2{}_2 - \tfrac{1}{2} XH_2J_3 + \tfrac{1}{2} ZJ_2J_3,$$

$$C \equiv HJ_2 + ZHJ_{23} + \tfrac{1}{2} XHJ_{33} - YJJ_{23} - \tfrac{1}{2} ZJJ_{33} - \tfrac{1}{2} H_3 + ZJ_{13}$$
$$+ \tfrac{1}{2} YJ_{33} - 2 J_1 - \tfrac{1}{2} ZJ_2H_3 + \tfrac{1}{2} YJ_2J_3 - \tfrac{1}{4} X J_3H_3 + \tfrac{1}{4} ZJ^2{}_3,$$

$$D \equiv -2ZHH_{22} - X HH_{23} + HJ_2 + 2YJH_{22} + ZJH_{23} - 2 ZH_{12}$$
$$+ \tfrac{1}{2} H_3 - YH_{23} + ZH^2{}_2 - YJ_2H_2 + \tfrac{1}{2} X H_2H_3 - \tfrac{1}{2} ZJ_2H_3,$$

$$E \equiv -HH_2 - ZHH_{23} - \tfrac{1}{2} XHH_{33} + \tfrac{1}{2} HJ_3 + YJH_{23} + \tfrac{1}{2} ZJH_{33}$$
$$- ZH_{13} - \tfrac{1}{2} YH_{33} + 2 H_1 + \tfrac{1}{2} ZH_2H_3 - \tfrac{1}{2} YH_2J_3 + \tfrac{1}{4} X H^2{}_3$$
$$- \tfrac{1}{4} ZJ_3H_3 - \tfrac{1}{2} JH_3.$$

Here the suffixes 1, 2, 3 denote differentiation with respect to X, Y, Z respectively.

We have, accordingly

$$f_{;\, i;\, j} = 2\, g_{ij} f_2 + 4\, \dot{x}_i\, \dot{x}_j f_{22} + p_i p_j f_{33} + 2\, (\dot{x}_i\, p_j + \dot{x}_j\, p_i)\, f_{23} \qquad (4)$$

The condition (2, *a*) for the given values of ρ_j^i (3) and $f_{;\, i;\, j}$ (4) leads to a linear partial differential equation of the second order for the function f :

$$2\,(C - D) f_2 + 4\,(CY + EZ)\, f_{22} + 2\,(CZ + EX - BY - DZ) f_{23}$$
$$- (BZ + DX)\, f_{33} = 0 \qquad (5, a)$$

The condition (2, *b*) gives the following set of equations:

$$- Jf_2 + 2\, Zf_{12} + 2\,(HZ - YJ) f_{22} + (Y + XH - ZJ) f_{23} = 0, \qquad (5, b)$$

$$2\, H_3 f_2 + (4\, H + 2\, ZH_3 - 2\, YJ_2) f_{23} + (XH_3 - J - ZJ_3) f_{33} + 2Zf_{331}$$
$$+ 2\,(ZH - YJ) f_{233} + (Y + XH - ZJ) f_{333} = 0, \qquad (5, c)$$

$$- J_2 f_2 + (2\, ZH_2 - 2YJ_2 - 3J) f_{22} + (1 + XH_2 - ZJ_2) f_{23} + 2\, Zf_{221}$$
$$+ (Y + XH - ZJ) f_{223} + 2\,(ZH - YJ) f_{222} = 0, \qquad (6, a)$$

$$(2H_2 - J_3) f_2 + (4\, H + 2\, ZH_3 - 2\, YJ_3) f_{22} + (-4\, J + 2\, ZH_2 - 2\, YJ_2$$
$$+ XH_3 - ZJ_3)\, f_{23}$$
$$+ (1 + XH_2 - ZJ_2) f_{33} + 4\, Zf_{123} + (2\, Y + 2\, XH - 2\, ZJ)\, f_{233}$$
$$+ 4\,(ZH - YJ) f_{223} = 0 \qquad (6, b)$$

Out of these five equations (5, *a*, *b*, *c*), 6 (*a*, *b*) we shall select those which are independent and of the lowest possible order. By ordinary calculation we find that (6, *a*) is obtained by differentiating (5, *b*) with respect to Y and, therefore, can be ignored. Again, differentiating (5, *b*) with respect to Z and subtracting twice of the differential equation from (6, *b*) we get a new equation of the second order :

$$(2\,H_2 + J_3) f_2 - 4 f_{21} - 2\,(ZH_3 - YJ_3) f_{22} + (-\,XH_3 + ZJ_3 + 2\,ZH_2 - 2\,YJ_2) f_{23} + (1 + XH_2 - ZJ_2) f_{23} = 0 \qquad (5, d)$$

?

So the system of partial differential equations for the unknown function f are (5, *a b*, *c*, *d*).

As an aid to integration we find two more differential equations got directly from Euler's equation :

$$\frac{d}{dt} f_{;i} - f_{;i} = 0 \qquad (7)$$

$$-\,Jf_2 + 2\,Zf_{12} + 2\,(HZ - YJ) f_{22} + (Y + XH - ZJ) f_{23} = 0, \qquad (7, a)$$

$$-\,2 f_1 + 2\,H f_2 + 2\,Z f_{13} + 2\,(ZH - YJ) f_{23} + (Y + XH - ZJ) f_{33} = 0 \qquad (7, b)$$

Equation (7, *a*) is the same as the equation (5, *b*) and (5, *c*) follows from (7, *b*) by differentiating it with respect to Z. But nothing is gained by taking (7, *b*) in place of any of the others because it is known that our set (5, *a–d*) would be necessary and sufficient for the existence of a metric. This seems a contradiction at first, which is easily explained by the well-known fact that one integrand of a variational problem can only be determined to an additive perfect differential. The solution of (7, *b*) can differ atmost by such an additional term from the set (5).

We know that our system of equations (5) admit a solution, for if we take $f = A(X)\,Z + B(X)$ all the equations are satisfied. But this is a trivial case which does not satisfy the condition (2, *c*). In order to get the most general non-trivial solution we must, in general, restrict the arbitrary functions H, J. Although our system is not of the linear first order partial differential equation type, we can nevertheless discuss the necessary conditions for the existence of a non-trivial metric by reduction to a linear first order system.

We can eliminate f_{33} from equations (5, *a*) and (5, *d*). This gives a new equation : ?

$$\{2\,(C-D)\,(1+XH_2-ZJ_2)+(BZ+DX)\,(2\,H_2+J_3)\}$$
$$f_2-4\,(BZ+DX)\,f_{21}$$
$$+\{4\,(CY+EZ)\,(1+XH_2-ZJ_2)-2\,(ZH_3-YJ_3)\,(BZ+DX)\}f_{22}$$
$$+\{2\,(CZ+EX-BY-DZ)\,(1+XH_2-ZJ_2)$$
$$+(BZ+DX)\,(ZJ_3+2\,ZH_2-XH_3-2\,YJ_2)\}\,f_{23}$$
$$=0 \qquad (8)$$

Now the equations (5, *b*) and (8) can be considered linear equations of the first order in $f_2 = \phi$ (say). Writing these equations in the following forms :

$$\nabla_1\phi \equiv \alpha_1\phi+\beta_1\phi_1+\gamma_1\phi_2+\delta_1\phi_3=0, \qquad (5, b)$$
$$\nabla_2\phi \equiv \alpha_2\phi+\beta_2\phi_2+\gamma_2\phi_2+\delta_2\phi_3=0, \qquad (8)$$

We calculate the Poisson's brackets[3] and get one more equation

$$\nabla_3\phi \equiv (\nabla_1\nabla_2-\nabla_2\nabla_1)\phi \equiv \alpha_3\phi+\beta_3\phi_1+\gamma_3\phi_2+\delta_3\phi_3=0. \qquad (9)$$

If α_3, β_3, δ_3, γ_3 all vanish, the system $\nabla_1\phi = \nabla_2\phi = 0$ is integrable ; if $\beta_3 = \gamma_3 = \delta_3 = 0$, $\alpha_3 \neq 0$, only trivial solution exists. In the general case we repeat the process once more and get,

$$\nabla_4\phi \equiv (\nabla_1\nabla_3-\nabla_3\nabla_1)\,\phi \equiv \alpha_4\phi+\beta_4\phi_1+\gamma_4\phi_2+\delta_4\,\phi_3=0, \qquad (10)$$
$$\nabla_5\phi \equiv (\nabla_2\nabla_3-\nabla_3\nabla_2)\,\phi \equiv \alpha_5\phi+\beta_5\phi_1+\gamma_5\phi_2+\delta\,\phi_3=0. \qquad (11)$$

Therefore, we get two necessary conditions for the existence of a non-trivial metric.

$$(a)\quad \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2' & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{vmatrix} = 0,$$

$$(b)\quad \begin{vmatrix} \alpha_4 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_5 & \beta_5 & \gamma_5 & \delta_5 \end{vmatrix} = 0. \qquad (12)$$

The process is not to be continued further because it is known that if (12) be satisfied, all further operators will be linearly dependent on ∇_1, ∇_2, ∇_3.

Conditions (12) are justified, because we could not with f a non-trivial solution get as many as four linearly independent operators $\nabla_r\phi = 0$ of the present type.

[3] T. Levi-Civita, *The Absolute Differential Calculus*, Ch. III.

If the equations (12) are satisfied, we can always integrate—in theory, though the actual process in practice need not be simple—for f_2, and have determined f, but for an additive arbitrary function of (X, Z). The additive function can be determined to within a perfect differential by substitution in the second order equation (5, *c*). It would have been pleasant to find the general existence conditions and solution explicitly, but this can be done only by transferring the equations to a less cumbersome set, hence is postponed for some future note.

It is to be noticed that there are just two Euler equations, and four in the system (5, *a–d*); yet each set is supposed to lead to the metric. The fact is that the equations (5, *b*) is common to both. Of the remaining three in the set (5), (5, *c*) and (5, *d*) can be eliminated from Euler equations as follows :

Differentiating (7, *b*) with respect to Z we get (5, *c*). Again differentiating (7, *a*) with respect to Z and (7, *b*) with respect to Y and subtracting the results so obtained we get equation (5, *d*). So, there only remains only one extra equation in (5), namely, (5, *a*) which cannot be deduced from the Euler equations. This is to be explained by well-known theoretical considerations.[2] Because equations (2) determine $f_{;\,i;\,j}$, f is undetermined in the great case only to within an additive term of type A (X) Z + B (X). This accounts for the difference in the two sets of equations. The best procedure would be to find an integral, if one exists, of our set $\nabla_i \phi = 0$, and substitute in the Euler equations to determine the additional terms.

II. There does not exist, in general, a non-trivial solution for the most general set of H, J. If the space admits a metric, such that any non-trivial function $\psi(f)$ is also a metric, the conditions (2) reduce to the following form :[4]

$$\mathfrak{D} f = 0, \ ; f_{|i} = 0, \text{ along with } |f_{;\,i;\,j}| \neq 0, \tag{13}$$

where | represents covariant differentiation in Kosambian sense.

Because the invariant metric (in Kosambi's terminology) f does not contain the parameter t, the above condition reduce to

$$\epsilon^r f_{;\,r} = 0 \ ; f_{|i} = 0, \tag{13, a, b}$$

where $\epsilon^r = \alpha^r - \frac{1}{2}\, \alpha^r_{;i}\, \dot{x}^i$,[5]

$$\alpha^r = -p^r \mathrm{H} + \dot{x}^r \mathrm{J}$$

[4] D. D. Kosambi, "Modern differential geometries," *Ind. Jour. Phys.*, 1932.

[5] ————, " Parallelism and Path-Spaces," *Math. Zeitschrift*, 1933, 37.

The two conditions lead to the following system of differential equations for f:

$$\nabla_1 f \equiv 2 f_1 + (ZH_3 - YJ_3) f_2 + (\tfrac{1}{2} XH_3 - \tfrac{1}{2} J - \tfrac{1}{2} ZJ_3) f_3 = 0 \qquad (14, a)$$

$$\nabla_2 f \equiv (2ZH_2 - J - 2YJ_2) f_2 + (1 + XH_2 - ZJ_2) f_3 = 0 \qquad (14, b)$$

$$\begin{aligned}\nabla_3 f \equiv & (-2\,ZH + YJ + 2\,YZH_2 + Z^2H_3 - 2\,Y^2J_2 - YZJ_3)\, f_2 \\ & + (-XH + XYH_2 + \tfrac{1}{2} XZH_3 - YZJ_2 - \tfrac{1}{2} Z^2J_3 + \tfrac{1}{2} ZJ) f_3 = 0 \qquad (14, c)\end{aligned}$$

From this set of equations we infer that the following conditions must be satisfied for the existence of a metric :

(i) the second order determinant formed out of the coefficients of f_2 and f_3 in (14, b), (14, c) must be zero, *i.e.*,

$$\begin{aligned}& -XJH + (2\,Z^2 - 2\,XY)\, HJ_2 + 2YZH_2 - 2Y^2J_2 + (2\,XY - Z^2)\, JH_2 \\ & - YZJJ_2 - 2ZH + YJ + Z^2H_3 - YZJ_3 + (Z^3 - XYZ)(J_3H_2 - J_2H_3) \qquad (15) \\ & + \tfrac{1}{2} XZJH_3 + \tfrac{1}{2} ZJ^2 - \tfrac{1}{2} Z^2JJ_3 = 0.\end{aligned}$$

This condition being satisfied, we proceed to find out a second condition for the existence of a metric.

From the compatibility conditions of (13) we know that

$$P_j^i f_{;\,i} = 0^5$$

Now, $P_j^i f_{;\,i} \equiv (2A\dot{x}_j + 2\,BY\dot{x}_j + 2\,CYp_j + 2\,DZ\dot{x}_j + 2\,E\,Zp_j) f_2 \qquad (16)$

$\qquad + (Ap_j + BZ\dot{x}_j + CZp_j + DX\dot{x}_j + EXp_j) f_3 = 0.$

From (16), equating the coefficients of $\dot{x}_j$, p_j we get two equations:

$$(2A + 2\,BY + 3\,DZ) f_2 + (BZ + DX) f_3 = 0 \qquad (16, a)$$

$$(2\,CY + 2\,EZ)\, f_2 + (A + CZ + EX)\, f_3 = 0, \qquad (16, b)$$

A, B, C, D, E as in formula (3).

The second condition that a metric should exist is that of the four equations, namely (14, b, c) and (16, a, b) each pair must be dependent.

If we take the paths as those with the most general Lorentz invariant symmetric affine connection, we have :

$$H = \frac{aY}{X} + \frac{bZ^2}{X^2},\ J = \frac{2\,cZ}{X}, \qquad (17)$$

a, b, c are functions of X only. In this case the condition (15) is satisfied. For the second condition we form the determinant out of the coefficients of equations (14, b), (16, a), and get after taking the common factor.

$$\begin{vmatrix} a - c & a + 1 \\ \left(\frac{bc}{X} - \frac{c^2}{X} + 2\,c' - \frac{2c}{X} + \frac{2b}{X} \right. & \left(-\frac{c}{X} - \frac{ac}{X} - 2a' + \frac{a}{X}\right. \\ \left. - 2\,a' + \frac{2a}{X} + \frac{a^2}{X} + \frac{ab}{X}\right) & \left. + \frac{b}{X} + \frac{a^2}{X} + \frac{ab}{X}\right) \end{vmatrix} = 0 \qquad (18)$$

$$\text{or, } X\left(\frac{a'}{1+a} - \frac{c'}{1+c}\right) = a + b - c \qquad (18, a)$$

The vanishing of the other determinants formed by taking any other pair gives the same result.

The equation (14, c) is identically satisfied for the values (17) specified to H, J.

If a, b, c are constant (18, a) gives $a + b = c$ the condition for the existence of a metric.*

Although it is learnt that (2, a, b) are always satisfied when $\mathfrak{D}f = f_{1,\,i} = 0$, it is not without interest to derive the set (5) from (14, a–c) in the following manner.

(1) Multiply (14, a) by Z, (14, b) by Y, add them. We get the equation

$$2\,Zf_1 + 2\,(ZH - YJ)\,f_2 + (XH + Y - ZJ)\,f_3 = 0 \qquad (19)$$

Differentiating (19) with respect to Y and substracting from the result (14, b) we get (5, b).

(2) Multiply (14, a) by 2 and differentiating with respect to Y and subtracting the result from the differentiated result of (14, b) with respect to Z we get (5, d).

(3) Differentiate (19) with respect to Z and subtract from the result twice of (14, a) we get (7, b) which differentiated with respect to Z gives (5, c).

The equation (5, a) cannot easily be deduced from (14) in the above manner. But if we proceed in a way parallel to the theory we can deduce it from the invariant form of equations (13).

* I am obliged to Prof. D. D. Kosambi for communicating to me these conditions (18, a). He had obtained them by a direct method, but has not yet seen fit to publish his results. His general condition for non-degeneracy $(a + 1)(c + 1) \neq 0$ has not been obtained here.

We know that if $\mathfrak{D} f = 0, f_{1\,i} = 0$, then

$$P_j^{\,i}\, f_{;\,i} = 0^5$$

Therefore differentiating (16) with respect to $\dot{x}^r$ we get,

$$
\begin{aligned}
&\{2\, A g_{rj} + 2 \dot{x}_j (2\, A_2 \dot{x}_r + A_3 p_r) + 2\, BY g_{rj} + 4\, B \dot{x}_j \dot{x}_r \\
&+ 2\, Y \dot{x}_j (2\, B_2 \dot{x}_r + B_3 p_r) + 4\, C p_j \dot{x}_r + 2\, Y p_j (2\, C_2 \dot{x}_r + C_3 p_r) \\
&+ 2\, D\, p_r \dot{x}_j + 2\, DZ g_{rj} + 2\, Z \dot{x}_j (2\, D_2 \dot{x}_r + D_3 p_r) \\
&+ 2\, E p_r p_j + 2\, Z p_j (2\, E_2 \dot{x}_r + E_3 p_r)\} f_2 + \\
&+ (2\, A \dot{x}_j + 2\, BY \dot{x}_j + 2\, CY p_j + 2\, ZD \dot{x}_j + 2\, ZE p_j) \\
&\qquad\qquad (2 f_{22} \dot{x}_r + f_{23} p_r) + \\
&+ \{p_j (2\, A_2 \dot{x}_r + A_3 p_r) + BZ g_{rj} + B p_r \dot{x}_j + Z \dot{x} (2\, B_2 \dot{x}_r + B_3 p_r) \\
&+ C p_r p_j + Z p_j (2\, C_2 \dot{x}_r + C_3 p_r) + DX g_{rj} + X \dot{x}_j (2\, D_2 \dot{x}_r + D_3 p_r) \\
&+ X p_j (2\, E_2 \dot{x}_r + E_3 p_r)\} f_3 \\
&+ (A p_j + BZ \dot{x}_j + CZ p_j + DX \dot{x}_j + EX p_j) \\
&\qquad\qquad (2\, f_{23} \dot{x}_r + f_{33} p_r) = 0
\end{aligned}
\tag{20}
$$

Again, $\frac{\partial}{\partial \dot{x}^j} (P_r^{\,i} f_{,i})$ can be found by interchanging j and r in equation (20)

$$
\begin{aligned}
&\therefore \frac{\partial}{\partial \dot{x}^r} (P_j^{\,i} f_{;\,r}) - \frac{\partial}{\partial \dot{x}^j} (P_r^{\,i} f_{;\,i}) \\
&= (\dot{x}_j p_r - \dot{x}_r p_j) \{(2\,(D - C) f_2 - 4\,(CY + EZ) f_{22} \\
&\qquad + (2BY + 2\,DZ - 2CZ - 2EX) f_{23} \\
&\qquad\qquad + (BZ + DX) f_{33}\} \\
&+ (\dot{x}_j p_r - \dot{x}_r p_j) \{(2A_3 + 2YB_3 - 2C - 4YC_2 + 2ZD_3 + 4ZE_2) f_2 \\
&\qquad + (-2\, A_2 + B + ZB_3 - 2ZC_2 + XD_3 \\
&\qquad\qquad - 2\, X\, E_2)\, f_3\}
\end{aligned}
$$

i.e., $0 =$ the left-hand side of equation $(5, a) + 3\, R_j{}^i{}_r f_{;\,i}$

But, $R_j{}^i{}_r\, f_{;\,i} = 0^5$ which follows from $f_{1i} = 0$.

Thus $(5, a)$ is deduced from (13).

CHAPTER V.

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EINSTEIN SPACES ADMITTING THE LORENTZ GROUP*

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I. IT was shown by Kosambi that the classical theories of relativity could be generalized by direct application of the Lorentz group to the trajectories. This gave rise to four-dimensional path-spaces, K_4

$$\ddot{x}^i - p^i \frac{Y}{X} G(\zeta) + 2\dot{x}^i \frac{Z}{X} \gamma(X, \zeta) = 0,^1 \tag{1}$$

$$(i = 0, 1, 2, 3;\ \gamma \text{ arbitrary}),$$

where Kosambi's notations are used:†

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \quad g_{ij} = 0 \quad i \neq j,$$
$$p^i = x^i, \quad p_j = g_{jr}p^r, \quad X = g_{ij}p^ip^j, \quad Y = g_{ij}\dot{x}^i\dot{x}^j,$$
$$Z = g_{ij}\,\dot{x}^ip^j, \ \xi = \frac{Z^2}{XY}.$$

It has also been shown by the same author in the form of a theorem that:

The most general path equations derivable from three-dimensional observations and admitting the Lorentz group are :

$$\ddot{x}^i - p^i\, Y\, G\left(X, \frac{Z^2}{Y}\right) + \dot{x}^i Z\gamma\left(X, \frac{Z^2}{Y}\right) = 0^2. \tag{2}$$

The purpose of this note is to discuss the conditions under which the most general Riemannian space, admitting the Lorentz group, can be an Einstein space when (i) the invariant R of the corresponding Ricci tensor survives, (ii) the invariant R vanishes.

In what follows we shall consider an n-dimensional Riemannian space because it has been pointed out by Kosambi[3] that if we extend our treatment to more than four-dimensions, with the corresponding ~~extended~~ Lorentz group, the results are valid also for n-dimensional space.

* It is a great pleasure to me to thank Prof. D. D. Kosambi for his kind suggestion and criticism during the course of this investigation. I have made free use of his unpublished manuscript entitled: 'Path-equations admitting the Lorentz group II'.

† Besides these notations I use the universal notations as given in Eisenhart's "Riemannian geometry".

II. The most general Riemannian metric, admitting the Lorentz group, has the form

$$f=\frac{\alpha Y}{X}+\frac{\beta Z^2}{X^2}=\left(\frac{\alpha g_{ij}}{X}+\frac{\beta p_i p_j}{X^2}\right)\dot{x}^i\dot{x}^j; \tag{3}$$

α, β are functions of X,

subject to the following conditions of non-degeneracy:

$$\left|\frac{\alpha g_{ij}}{X}+\frac{\beta p_i p_j}{X^2}\right|=-\frac{\alpha^3(\alpha+\beta)}{X^4}\neq 0^3 \tag{4}$$

If we put

$$\frac{\alpha'}{\alpha+\beta}-1=A,\ \frac{2\alpha'\beta-\alpha\beta'}{\alpha(\alpha+\beta)}=B,\ \frac{\alpha'}{\alpha}-1=C, \tag{5}$$

where *dash* indicates differentiation with respect to *log X*, the geodesics of (3) are given by

$$\ddot{x}^i-p^i\left(\frac{AY}{X}+\frac{BZ^2}{X^2}\right)+2C\dot{x}^i\frac{Z}{X}=0^3 \tag{6}$$

It is obvious that equations (6) are the particular cases of equations (1) and (2).

Taking
$$\bar{g}_{ij}=\frac{\alpha g_{ij}}{X}+\frac{\beta p_i p_j}{X^2}$$
to be the fundamental metric tensor, we get the following invariants (the bar, put above, indicates that these invariants are derived from $\bar{g}_{ij}$ as the fundamental tensor).

$$\bar{R}^i_{jkl}=\frac{U}{\alpha\gamma X}(\delta^i_l g_{jk}-\delta^i_k g_{jl})+\frac{S}{\alpha^2\gamma X^2}(\delta^i_k p_j p_l-\delta^i_l p_j p_k)$$
$$+\frac{Q}{\alpha\gamma^2 X^2}(p^i p_l g_{jk}-p^i p_k g_{jl}), \tag{7, a}$$

$$\bar{R}_{ijkl}=\frac{U}{\gamma X^2}(g_{il}g_{jk}-g_{ik}g_{jl})+\frac{Q}{\alpha\gamma X^3}(p_i p_l g_{jk}-p_i p_k g_{jl})$$
$$+\frac{(\gamma-\alpha)\,U}{\alpha\gamma X^3}(p_i p_l g_{jk}-p_i p_k g_{jl}$$
$$+\frac{S}{\alpha\gamma X^3}(g_{ik}p_j p_l-g_{il}p_j p_k), \tag{7, b}$$

$$\bar{R}_{ij}=\frac{g_{ij}}{X}\left\{(n-1)\frac{U}{\alpha\gamma}+\frac{Q}{\alpha\gamma^2}\right\}-\frac{p_i p_j}{X^2}\left\{(n-1)\frac{S}{\alpha^2\gamma}+\frac{Q}{\alpha\gamma^2}\right\}, \tag{8, a}$$

$$\bar{R}_{ij}=\bar{g}_{ij}\{(n-1)U\gamma+Q\}\frac{1}{\alpha^2\gamma^2}+p_i p_j\frac{(n-2)Q}{\alpha^2\gamma X^2}, \tag{8, b}$$

$$\overline{R} = (n-1)\,(nU\gamma + 2Q)\,\frac{1}{\alpha^2\gamma^2}, \tag{9}$$

$$X^2\overline{W}^i{}_{jkl} = \frac{Q}{\alpha\gamma^2}\left\{\frac{X}{n-1}\,(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk}) + p^i\,(p_l g_{jk} - p_k g_{jl}) + \frac{1}{n-1}\,(\delta^i{}_l p_j p_k - \delta^i{}_k p_j p_l)\right\}, \tag{10}$$

$$\overline{C}_{ijkl} = 0, \tag{11}$$

where,

$$\left.\begin{aligned} &\alpha + \beta = \gamma \\ &U = \alpha'^2 - \alpha\,\gamma \\ &Q = 2\,\alpha''\alpha\,\gamma - \alpha\,\alpha'\gamma' - 2\,\alpha'^2\,\gamma + \alpha\,\gamma^2 \\ &S = -2\,\alpha''\alpha\,\gamma + \alpha\,\alpha'\gamma' + \alpha\,\alpha'^2 - \alpha^2\gamma + \alpha'^2\gamma \end{aligned}\right\} \tag{12}$$

The vanishing of the conformal curvature tensor (11) gives us:

THEOREM I: *The Riemannian spaces, admitting the Lorentz group, can be mapped conformally on an S_n for* $n \geqslant 4$.

This theorem indicates that a transformation of co-ordinates exists such that

$$\frac{\alpha Y}{X} + \frac{\beta Z^2}{X^2} \to \overline{Y}\psi\,(\overline{X}). \tag{13}$$

Let such a transformation be given by

$$\overline{x}^i = \phi\,(X)\,x^i \tag{14}$$

Then,
$$\overline{Y} = \phi^2\,Y + 4\,Z^2\,(\phi\,\phi' + \phi'^2\,X). \tag{15}$$

In order that (13) be satisfied it is necessary and sufficient that

$$\frac{4\,(\phi\phi' + \phi'^2 X)}{\phi^2} = \frac{\beta}{\alpha X}. \tag{16}$$

The equation (16), being of standard form, can be integrated.

III. An n-dimensional Einstein space is one for which

$$\overline{R}_{ij} = \frac{\overline{R}}{n}\,\overline{g}_{ij}. \tag{17}$$

Substituting the values of R_{ij} and $\overline{R}$, (17) becomes

$$\overline{g}_{ij}\,(2-n)\,\frac{Q}{4\alpha^2\gamma^2} + p_i p_j\,(n-2)\,\frac{Q}{\alpha^2\gamma X^2} = 0. \tag{18}$$

The necessary and sufficient condition that the metric (3) should satisfy equations (17) is that

$$Q = 0. \tag{19}$$

Whence, $$\overline{W}^{i}_{jkl}=0. \tag{20}$$

The vanishing of the projective curvature tensor indicates also the isotropy of the space.[4]

This gives us:

THEOREM II: *Einstein spaces, admitting the Lorentz group, are projectively flat and hence isotropic.*

In classical Riemannian geometry it has been shown by Schouten and Struik[5] that 'an Einstein space, if conformal to a flat space, is isotropic'. But in the present case, under discussion, an Einstein space, admitting the Lorentz group, is isotropic as well as conformal to a flat space. Therefore Theorem II is a consequence of Theorem I, hence the result of Schouten Struik.

Because the isotropic spaces of positive curvature are of class one[6], we shall find the condition such that Einstein spaces, admitting the Lorentz group, can be of positive curvature.

For $Q=0$, the curvature tensor $(7, b)$ takes the following form:

$$\overline{R}_{ijkl}=-\frac{U}{\alpha^2\gamma}(\bar{g}_{ik}\bar{g}_{jl}-\bar{g}_{il}\bar{g}_{jk}). \tag{21}$$

Putting (19) in the following form

$$U'-U\left(\frac{2\alpha'}{\alpha}+\frac{\gamma'}{\gamma}\right)=0, \tag{22}$$

it follows on its integration that

$$\frac{U}{\alpha^2\gamma}=\text{constant}=k^2 \text{ (say)}. \tag{23}$$

In order that the Einstein space, admitting the Lorentz group, be of constant positive curvature.

$$\left.\begin{aligned}&-\frac{U}{\alpha^2\gamma}>0\\ &\text{or, } \alpha\gamma-\alpha'^2>0\end{aligned}\right\} \tag{24}$$

Because of the physical interest, as will be shown in Sec. IV, we have a result for the most interesting case $n=4$.

It is possible to represent a natural gravitational field in flat spaces of five dimensions, if and only if $\alpha\gamma>\alpha'^2$.

If we take

$$\overline{R}\equiv(n-1)\frac{1}{\alpha^2\gamma^2}(nU\gamma+2Q)=0, \tag{25}$$

equations (17) become

$$\bar{g}_{ij}\,\{(n-1)\,U\gamma+Q\}\,\frac{1}{\alpha^2\gamma^2}+\frac{(n-2)\,Q}{\alpha^2\gamma\,X^2}\,p_i p_j=0. \tag{26}$$

Now, the necessary and sufficient condition that the metric (3) should satisfy equations (17) is that

$$\left.\begin{aligned}&(i)\ Q=0,\\&(ii)\ (n-1)\,U\gamma+Q=0.\end{aligned}\right\} \tag{27}$$

The equations (25, 27) are compatible and the common solution is given by

$$U\equiv\alpha'^2-\alpha\,\gamma=0, \tag{28}$$

which ~~equivalent~~ of a condition due to Kosambi.[3]

Equation (28) mkes $R^i_{jkl}\equiv 0$, for in (7, *a*) Q and S can be put in the form of U and its derivative with respect to *log X*. Therefore We have

THEOREM III: *Einstein spaces, admitting the Lorentz group, are flat, if and only if* $\bar{R}=0$.

Further integration of (28) is not possible unless we assign definite values to γ.

IV. The results, obtained in the previous sections, hold good also for $n=4$. Therefore, in the following pages we shall consider their physical aspects.

A fundamental need in Milne's theory of the expanding universe is that $\dot{x}^i=c\,x^i$ be a solution of equations (6), which leads to the condition that

$$\left.\begin{aligned}&A+B=2\,C+1^2\\&\textit{i.e.},\quad \gamma=\text{constant}\neq 0.\end{aligned}\right\} \tag{29}$$

Now, the equations (23, 28) can be integrated; their solutions being respectively

$$\alpha=\tfrac{1}{2}\,m^k\,X^{2bk}+\frac{1}{8\,k^4\,m^k\,X^{2bk}}-\frac{1}{2\,k^2}, \tag{30}$$

where m, k^2 are constants of integration and $\gamma=4\,b^2$.

$$\alpha=(a+b\log X)^2, \tag{31}$$

where a is constant of integration and $\gamma=4\,b^2$.

So far our investigation was limited to pure geometrical considerations without any application of the Einstein-field equations.

$$\bar{R}_{ij}-\tfrac{1}{2}\,\bar{R}\bar{g}_{ij}=-8\,\pi\bar{T}_{ij}, \tag{32}$$

when matter is present in the space. Equations (32) reduce to $\bar{R}_{ij}=0$ when he space is empty and this case, which corresponds to Schwarzscheld's classical treatment, has been discussed in Sec. III. It is to be noted that the Schwarzscheld values of g_{ij} for empty space do not make the space flat, except at a very great distance from the centre of mass, but in the present case the vanishing of $\bar{R}_{ij}$ along with $\bar{R}=0$ gives a flat space (Theorem III).

Substituting the values of $\bar{R}_{ij}$, R into (32) we get the field-equations as follows:

$$\bar{g}_{ij}\left(-4\,\alpha''\alpha\gamma+\alpha'^2\gamma+2\alpha\alpha'\gamma'+\alpha\gamma^2\right)\frac{1}{\alpha^2\gamma^2}$$

$$+\frac{p_i p_j}{X^2}\cdot\frac{2}{\alpha^2\gamma}\left(2\,\alpha''\alpha\gamma-\alpha\alpha'\gamma'-2\alpha'^2\gamma+\alpha\gamma^2\right)=-8\,\pi\,\bar{T}_{ij}. \quad (33)$$

But

$$\bar{T}_{ij}=\bar{\epsilon}\lambda_i\lambda_j-p\,\bar{g}_{ij}{}^6 \quad (34)$$

$$=\bar{\epsilon}\,\dot{x}_i\dot{x}_j\,M-p\bar{g}_{ij}{}^2$$

where,

$$M=\frac{\left\{\dfrac{\alpha^2}{X^2}+\dfrac{2\,\alpha\,(\gamma-\alpha)\,Z}{X^3}\sqrt{\dfrac{X}{Y}}+\dfrac{(\gamma-\alpha)^2Z^2}{X^3Y}\right\}}{\dfrac{\alpha Y}{X}+(\gamma-\alpha)\dfrac{Z^2}{X^2}}. \quad (35)$$

$$\therefore\ p=\frac{1}{8\,\pi\,\alpha^2\gamma^2}\left(-4\,\alpha''\alpha\,\gamma+2\,\alpha\,\alpha'\gamma'+\alpha'^2\,\gamma+\alpha\,\gamma^2\right)$$

$$\bar{\epsilon}=-\frac{1}{4\pi\alpha^2\gamma XY}\left(2\,\alpha''\alpha\gamma-\alpha\alpha'\gamma'-2\alpha'^2\,\gamma+\alpha\gamma^2\right),$$

for

$$p_i p_j=\dot{x}_i\dot{x}_j\frac{X^2}{Y}. \quad (36)$$

If $\bar{\epsilon}=0,\ p=\dfrac{3}{8\,\pi\,\alpha^2\gamma}\left(\alpha\,\gamma-\alpha'^2\right)$

Therefore, if density is zero, pressure is +ve, zero, −ve, according as

$$\alpha\,\gamma \gtreqless \alpha'^2 \quad (37)$$

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CHAPTER VI.

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EQUATIONS OF KILLING FOR LORENTZ-INVARIANT RIEMANN SPACES

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Here I attempt to investigate the solutions of the equations of Killing for the Riemannian spaces admitting the Lorentz-group, L_6, and study the properties of the groups of motions associated with the various such spaces as arise in the course of our investigation.

The metric of a Riemann space admitting the Lorentz-group has the form[1]

$$F = \left[\frac{A(X)}{X} a_{ij} + \frac{B(X)}{X^2} p_i p_j\right] \dot{x}^i \dot{x}^j \tag{1}$$

where,

$$\left.\begin{array}{l} a_{11} = -a_{22} = -a_{33} = \ldots\ldots = 1, \\ a_{ij} = 0,\ i \neq j,\ p^i = x^i, \\ X = a_{ij}\, p^i p^j,\ p_r = a_{ir}\, p^i,\ Y = a_{ij}\, \dot{x}^i \dot{x}^j. \end{array}\right\} \tag{2}$$

But these spaces being conformal to a flat space, the metric tensor can always be put in the canonical form[2]

$$g_{ij} = \frac{\alpha(X)}{X} a_{ij}. \tag{3}$$

The equations of Killing, $u_{i|j} + u_{j|i} = 0$, for (3) are of the form

$$\frac{\alpha}{X}(a_{rj}\, u^r_{,i} + a_{ir}\, u^r_{,j}) + 2\, a^r p_r \frac{\alpha' - \alpha}{X^2} a_{ij} = 0, \tag{4}$$

where,

$$\alpha' = \frac{d\alpha}{d \log X}.$$

Equations (4) are resolved into the following equations for different values of i, j $(i, j = 1, 2, 3, 4)$:

$$\left.\begin{array}{l} u^1_{,1} = u^2_{,2} = u^3_{,3} = u^4_{,4} = u^r p_r \dfrac{1}{X}\left(1 - \dfrac{\alpha'}{\alpha}\right); \\ u^1_{,2} = u^2_{,1};\ u^1_{,3} = u^3_{,1};\ u^1_{,4} = u^4_{,1};\ u^2_{,3} = -u^3_{,2}; \\ u^2_{,4} = -u^4_{,2};\ u^3_{,4} = -u^4_{,3}. \end{array}\right\} \tag{5}$$

The most general analytic solution of equations (5) may be taken in the form

$$u^i = a^i_{hk}(X)\, x^h x^k + K^i_h(X)\, x^h + h^i(X) \tag{6}$$

because the parameters corresponding to the higher degree terms vanish identically.

Substituting these values in (5) we get

$$\frac{1}{X}\left(1-\frac{a'}{a}\right)(A\,X\,x^1+B\,X\,x^2+C\,X\,x^3+D\,X\,x^4+g\,X+h^1x^1-h^2x^2-h^3x^3-h^4x^4) \\ =2\,(Ax^1+Bx^2+Cx^3+Dx^4)+g, \tag{7}$$

with the following relations :

$$\left.\begin{aligned} &K^1_2=K^2_1=a;\ K^1_3=K^3_1=b;\ K^2_3=-K^3_2=c\ ;\ K^1_4=K^4_1=d\ ;\\ &K^2_4=-K^4_2=e;\ K^3_4=-K^4_3=f\ ;\ K^1_1=K^2_2=K^3_3=K^4_4=g\ ;\\ &a^1_{11}=a^2_{12}=a^3_{13}=a^4_{14}=a^1_{22}=a^1_{33}=a^1_{44}=A\ ;\\ &a^1_{12}=a^2_{22}=a^3_{23}=a^4_{24}=a^2_{11}=-a^2_{33}=-a^2_{44}=B\ ;\\ &a^1_{13}=a^2_{23}=a^3_{33}=a^4_{34}=a^3_{11}=-a^3_{22}=-a^3_{44}=C\ ;\\ &a^1_{14}=a^2_{24}=a^3_{34}=a^4_{44}=a^4_{11}=-a^4_{22}=-a^4_{33}=D. \end{aligned}\right\} \tag{8}$$

The remaining components of a^i_{hk} are zero in virtue of the equations of Killing.

The six parameters $(a, b, \ldots\ldots, f)$ of (8) are absent in equation (7) and this is expected because these are precisely the six parameters of the Lorentz-group, L_6, which all our spaces admit. Our object is to find the values of α (X) from equation (7) such that the space admits, if possible, groups with one, two, three, and four more parameters besides L_6. For we know from classical results[3] that there are at most 10 parameters for any four-dimensional Riemann space, and this maximum number only in case of isotropic spaces.

There are only two restrictions on α which we can deduce from equation (7), taken as identities in x^1, x^2, x^3 and x^4, in order to determine the solutions of the equations of Killing involving more than six parameters.

(1) $$A=B=C=D=0\ ;\ h^1=h^2=h^3=h^4=0,$$

and therefore

$$1-\frac{a'}{a}=1.$$

Then, $\alpha=$ const. $=$ K (say).

Thus the space with metric $\frac{KY}{X}$ (Milne's Kinematic Case) admits two distinct groups with a total of 7 parameters $G_7 : (L_6+S_1)$,[4]

and

$$u^r=gx^r,$$

which gives the similitude group, S_1.

(2) The only other possibility is given by the following equations :

$$\left.\begin{aligned} &g=0,\\ &f\,(AX+h^1)=2AX,\\ &f\,(BX-h^2)=2BX,\\ &f\,(CX-h^3)=2CX,\\ &\text{and } f\,(DX-h^4)=2\,DX, \text{ where } f=1-\frac{a'}{a}\cdot \end{aligned}\right\} \tag{9}$$

All eight parameters (A, B, C, D, h^1, h^2, h^3, h^4) cannot be distinct for there are at most 4 more parameters in addition to those of L_6. Therefore, three possibilities arise :

(*a*) $$A = B = C = D = 0$$

$$\text{Then } f = 1 - \frac{\alpha'}{\alpha} = 0$$

$$\therefore\ \alpha = CX, \text{ (inasmuch as } \alpha' = d\alpha/d \log X).$$

The flat space CY (Einstein's Galilean space of special relativity) admits group of motions with 10 parameters $G_{10} : (L_6 + T_4)$. These additional parameters h^i yield the translation group T_4.

(*b*) $$h^1 = h^2 = h^3 = h^4 = 0.$$

$$\text{Then, } f - 2 = 0, \text{ i.e., } 1 - \frac{\alpha'}{\alpha} - 2 = 0$$

$$\therefore\ \alpha = \frac{C}{X}$$

i.e., the only other flat space of our type $\frac{CY}{X^2}$ admits group of motions with 10 parameter, G'_{10}: $(L_6 + I_4)$, though the I_4 defined by A, B, C, D is not a translation group in the co-ordinate chosen.

The four vector generators of I_4 are given by

(A) :—$u^i = [2x^{1^2} - X\ ,\ 2x^1x^2,\ 2x^1x^3,\ 2x^1x^4]$,

(B) :—$u^i = [2x^1x^2,\ X + 2x^{2^2},\ 2x^2x^3,\ 2x^2x^4]$,

(C) :—$u^i = [2x^1x^3,\ 2x^2x^3,\ X + 2x^{3^2},\ 2x^3x^4]$, and

(D) :—$u^i = [2x^1x^4,\ 2x^2x^4,\ 2x^3x^4,\ X + 2x^{4^2}]$.

(*c*) More generally the four equations (9) reduce to a single one

$$X(f - 2) = qf, \text{ where } q \text{ is a const.} \tag{10·1}$$

$$\therefore\ \alpha = \frac{CX}{(q - X)^2}. \tag{10·2}$$

If we put $q = \frac{K_0}{4}$, then (10·2) becomes

$$\alpha = \frac{16\ CX}{(K_0 - 4X)^2}, \tag{10·3}$$

which gives the metric of the isotropic space with its curvature K_0.

The four vector generators u^i, which are not generators of a 4-parameter group, are given by

(A′) :—$u^i = \left[2x^{1^2} - X - \frac{K_0}{4},\ 2x^1x^2,\ 2x^1x^3,\ 2x^1x^4\right]$,

(B′) :—$u^i = \left[2x^1x^2,\ X + 2x^{2^2} + \frac{K_0}{4},\ 2x^2x^3,\ 2x^2x^4)\right]$,

(C′) :—$u^i = \left[2x^1x^3,\ 2x^2x^3,\ X + 2x^{3^2} + \frac{K_0}{4},\ 2x^3x^4\right]$, and

(D′) :—$u^i = \left[2x^1x^4,\ 2x^2x^4,\ 2x^3x^4,\ X + 2x^{4^2} + \frac{K_0}{4}\right]$.

Thus we get the well-known result, *viz.*,

The isotropic spaces admit the groups of motions with 10 parameters, $G''_{10} = [L_6 + I'_4]$; *here* I'_4 *is not a 4-parameters group.*

There exists one more solution of the equation

$$a'^2 - a^2 - K_0 a^3 = 0^{(2)}$$

which determines the isotropy of the space $\frac{aY}{X}$. This further solution is given by

$$a = \frac{16\,CX}{(K_0 X - 4)^2}. \tag{10·4}$$

But this can be obtained from (10·3) by the transformation of co-ordinates given by

$$\bar{x}^i = \frac{x^i}{X}, \tag{10·5}$$

which is also the transformation connecting the two flat spaces.[5]

We sum up the results as follows :

(i) a unrestricted admits L_6 of which, as already known[6], there are two invariant sub-groups.

(ii) a = const. admits $G_7 = L_6 + S_1$ of which L_6 is an *invariant sub-group.*

(iii) $a = X$ admits $G_{10} = L_6 + T_4$ of which T_4 is *Abelian and the only invariant sub-group.*

(iv) $a = \frac{1}{X}$ admits $G'_{10} = L_6 + I_4$ of which I_4 is *Abelian and the only invariant sub-group.*

(iii) and (iv), moreover, are not two distinct cases, but the same, the co-ordinates being different.

(v) $a = \frac{16\,CX}{(K_0 - 4X)^2}$ admits $G''_{10} = L_6 + I'_4$, but the vectors constituting I'_4 are not generators of a group themselves. This G''_{10} is seen to be *simple* because the four infinitesimal transformations constituting I'_4 do not form a group but when alternated generate L_6, thus G''_{10} has the same structure as the *projective group* of the hypersurface of the second degree.[6]

I express my thanks to Prof. D. D. Kosambi for his kind suggestions and guidance.

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CHAPTER VII.

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ON THE CLASSIFICATION OF THE LORENTZ-INVARIANT SPACES ACCORDING TO THE GROUPS THEY ADMIT

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The purpose of the present work is the classification of the most general path-spaces with a symmetric affine connection admitting the Lorentz-group, L_6, according to the *possible* groups of *affine collineations* and *local applicabilities* which they admit. It has been shown in this note that the path-spaces, under discussion, fall into two main classes—(*a*) one for which the space admits a metric and (*b*) the other for which it does not admit any metric.

The structures of the groups arising in the course of our investigation have also been discussed here.

The methods and notations in the present paper are the same as those used in my previous paper.[1]

I. Groups of Collineations :

The most general path-spaces with a symmetric affine connection admitting the Lorentz-group have already been put by Kosambi[2] in the form

$$\ddot{x}^i - p^i \left\{ A(X) \frac{Y}{X} + B(X) \frac{Z^2}{X^2} \right\} + 2\dot{x}^i C(X) \frac{Z}{X} = 0. \tag{1}$$

But for the sake of simplicity of calculations we shall reduce the equations (1) to the canonical form

$$\ddot{\bar{x}}^i - \bar{p}^i P(\bar{X}) \frac{\bar{Y}}{\bar{X}} + 2\dot{\bar{x}}^i Q(\bar{X}) \frac{\bar{Z}}{\bar{X}} = 0, \tag{2}$$

by the transformation of co-ordinates $\bar{x}^i = x^i \psi(X)$, the condition for the non-singularity of the transformation being

$$\left| \frac{\partial \bar{x}^i}{\partial x^j} \right| \equiv \psi^3 (\psi + 2\psi') \neq 0, \; (n = 4)$$

where

$$\psi' = \frac{d\psi}{d \log X}.$$

Hereafter we shall drop the bars put above the equations (2).

That such a transformation is possible, can be shown by actually finding the value of ψ in terms of the given functions A, B, C in equations (1).

We get the following relations between (A, B, C), (P, Q) and ψ :

$$\left.\begin{aligned} &P\psi - 2\psi' = A(\psi + 2\psi'),\\ &Q(\psi + 2\psi') + 2\psi' = C\psi,\\ \text{and } &-4\psi'' + 4\psi' + 4P\psi' - 4Q\psi' - 4P\frac{\psi'^2}{\psi} - 8Q\frac{\psi'^2}{\psi} = B(\psi + 2\psi') - 4C\psi'. \end{aligned}\right\} \tag{3}$$

Eliminating P, Q from (3) we get

$$-4\psi'' + 4\psi' + 4\left(\psi' + \frac{\psi'^2}{\psi}\right)\left(A + \frac{2\psi'}{\psi}\overline{A+1}\right) + 8\frac{\psi'^2}{\psi} = B(\psi + 2\psi') \tag{4}$$

from which

$$\psi = L \exp.\left(\int u d \log X\right),$$

where

$$(2u+1)^2 = \frac{4\exp.\left[-\int(A+B+1)\, d\log X\right]}{K - 4\int\{(A+1)\exp.\left[-\int(A+B+1)\, d\log X\right]\}\, d\log X}.$$

Here K and L are constants of integration.

Eliminating ψ and its derivatives from (3) we have :

$$\left.\begin{aligned} P + Q + PQ &= A + C + AC\\ \text{and } P^2 + 2P - 2P' &= A^2 + 2A - 2A' + B + AB. \end{aligned}\right\} \tag{5}$$

Also

$$\frac{P'}{P+1} - \frac{Q'}{Q+1} - P + Q = \left(\frac{A'}{A+1} - \frac{C'}{C+1} - A - B + C\right)\frac{\psi^4}{\left|\frac{\partial x^i}{\partial x^j}\right|}, \tag{6}$$

which shows that the condition for the existence of a metric[2] in our new system of co-ordinates is

$$\frac{P'}{P+1} - \frac{Q'}{Q+1} = P - Q.$$

The equations

$$u^i{}_{|j|k} + u^r B^i_{jrk} = 0, \tag{7, a}$$

whose solutions determine the groups of affine collineations,[3] take the following form for our path-equations (2) :

$$\left.\begin{aligned} &u^i_{,j,k} - \frac{P}{X}p^i(u^h_{,k}a_{hj} + u^h_{,j}a_{hk}) + \frac{Q}{X}(\delta^i_j u^h_{,k} + \delta^i_k u^h_{,j})p_h\\ &+\frac{P}{X}a_{jk}p^h u^i_{,h} - \frac{P}{X}u^i a_{,jk} - \frac{2}{X^2}(P' - P)u^h p_h p^i a_{jk}\\ &+\frac{2}{X^2}(Q' - Q)u^h p_h(\delta^i_j p_k + \delta^i_k p_j) + \frac{Q}{X}u^h(\delta^i_j a_{hk} + \delta^i_k a_{hj}) = 0. \end{aligned}\right\} \tag{7, b}$$

The most general *analytic* solution of equations (7, *b*) may be taken in the form

$$u^i = h^i(X) + a^i_j(X)x^j + b^i_{jk}(X)x^j x^k + \ldots\ldots \tag{8}$$

Substituting (8) in (7, *b*) we determine the values of the surviving coefficients h^i, a^i_j, b^i_{jk}, etc., for the various spaces which are classified below, and observe that there is no need going beyond the second degree terms in (8). For all the coefficients in higher degree terms are identically zero *except* in the case of the most general flat spaces ——— (i) $P + Q + PQ = 0$, generate continuous groups of transformations in a space of n dimensions when, and only when, the complete set of solutions be finite in number.

If $u^i = \lambda^i, \mu^i, \ldots\ldots$, etc., are the solutions of (10), we call $\lambda^i \frac{\partial f}{\partial x^i}, \mu^i \frac{\partial f}{\partial x^i}, \ldots\ldots\ldots$ etc., the generators of the local applicabilities. Furthermore, we establish a theorem analogous to that given by Eisenhart[3] in connection with the groups of collineations :

If $\lambda^i \frac{\partial f}{\partial x^i}, \mu^i \frac{\partial f}{\partial x^i}, \ldots\ldots$ etc., are the generators of local applicabilities, so also are the alternants

$$[\lambda, \mu]^i \frac{\partial f}{\partial x^i} \equiv (\lambda^r \mu_{|r} - \mu^r \lambda^i_{|r}) \frac{\partial f}{\partial x^i}.$$

Differentiating the equations (10) covariantly w.r.t.x^h, substituting μ^i and λ^i for u^i, contracting the resulting equations with λ^h and μ^h separately and subtracting the equations contracted with μ^h from the equations contracted with λ^h we have, by making use of the Ricci-identities, the equations

$$[\lambda, \mu]^r B^i_{jkl\,|\,r} + [\lambda, \mu]^r_{\ |j} B^i_{rkl} + [\lambda, \mu]^r_{\ |k} B^i_{jrl} + [\lambda, \mu]^r_{\ |l} B^i_{jkr} - [\lambda, \mu]^i_{\ |r} B^r_{jkl} = 0 \tag{11}$$

which shows that the alternants are also the solutions of the equations $SB^i_{jkl} = 0$.

Suppose that a given space admits r independent solutions of the equations (10). Then, if r be finite, we have

$$(D_\alpha, D_\beta) f = C^\gamma_{\alpha\beta} D_\gamma f$$

where $D_\alpha f = u^i_{(\alpha)} \frac{\partial f}{\partial x^i}$ and $C^\gamma_{\alpha\beta}$ are constants.

Hence as a consequence of the fundamental theorem of the theory of continuous groups we have :

When, and only when, equations (10) *admit* r *independent solutions*, r *being finite, the space admits an* r*-parameter continuous group of local applicabilities.*

Following the similar procedure as in Sec. I we get the same classification of the path-spaces as regards the groups of local applicabilities as those investigated in Sec. I with the only following exceptions :

(i) *Every flat space is always applicable to itself for all values of* u^i.—We may say that the flat space admits *every* finite continuous group of applicabilities ; the totality of these finite groups, along with their commutators, given rise to G_∞ which is not a group in the sense of Lie.

(ii) The metric spaces for which

$$P + Q + PQ = 2Q + Q^2 - 2Q' = 2P + P^2 - 2P' = \text{const.}$$

admit $p^i \phi(X)$ as the vector generators of a one-parameter group, ϕ being arbitrary function of X. Therefore, the totality of the ϕ's form a group which might be called M_∞. This M_∞ is not a group in the sense of Lie.

We shall now investigate the *possible* Lie sub-groups (of finite order) of M_∞ by taking a set of constants of structure, and solving the equations arising therefrom, that is, by taking

$$D_1 = x^i \frac{\partial}{\partial x^i} \text{ and } D_2 = \phi\, x^i \frac{\partial}{\partial x^i}$$

(ii) $2P' - P^2 - 2P = 0$, $(P \neq 0,\ Q \neq 0)$. The solutions for such flat spaces involve the terms of the third degree, but can also be determined by transforming the solutions obtained for the flat space, $(P = Q = 0)$, by a suitable transformation of co-ordinates which can be determined from equations (3, 4). Hence we omit the third degree terms even for the determination of the solutions for the most general flat spaces mentioned above.

Classification:

(i) Einstein's "Galilean" space of the special theory of relativity admits 20 independent solutions corresponding to 20 parameters and are given by

$$\begin{aligned} u^i = [&gx^1 + ax^2 + bx^3 + cx^4 + h',\ (a + E)\, x^1 + (g + \lambda)\, x^2 + dx^3 + ex^4 + h^2, \\ &(b + F)\, x^1 - (d + G)\, x^2 + (g + \mu)\, x^3 + fx^4 + h^3, \\ &(c + H)\, x^1 - (e + I)\, x^2 - (f + J)\, x^3 + (g + \nu)\, x^4 + h^4]. \end{aligned} \tag{9}$$

These solutions generate a group $G_{20} = L_6$ (corresponding to the parameters a, b, c, d, e, f) $+ T_4\ (h^i) + \Gamma_{10}\ (E, F, G, H, I, J, g, \lambda, \mu, \nu)$ of which T_4 (translation) is its only *invariant sub-group*. We see that Γ_{10} is an extra group which does not occur in the group of motions for this special space. Considering Γ_{10} to be the direct product of A_9 $(E, F, G, H, I, J, \lambda, \mu, \nu)$ and S_1 (g) we see that A_9 is a *maximum* invariant sub-group of Γ_{10}. Besides this A_9 there are three more *maximum* invariant sub-groups of Γ_{10}, namely those wherein g is substituted in A_9 for each λ, μ, ν.

The other flat spaces, besides the one discussed above, also admit $\overline{G}_{20}$ of collineations whose vector generators are given by transforming (9) by the *proper* transformation of co-ordinates. The structure and properties of the groups $\overline{G}_{20}$ remain invariant under the transformation of co-ordinates.

(ii) The space for which $P = \text{const.} = \lambda$, $Q = \text{const.} = \mu$ admits $G_7 = L_6 + S_1$ of collineations of which L_6 and the similitude group S_1 are both invariant sub-groups. The generator of S_1 is given by $x^i \dfrac{\partial}{\partial x^i}$. The metric space follows when $\lambda = \mu$. In the particular case, when $\lambda = \mu = -1$, the space admits $G_8 = L_6 + S_1 + R_1$ of collineations of which $G_7 = L_6 + S_1$ is the only *maximum* invariant sub-group. The generator of R_1 is given by $\log X\, x^i \dfrac{\partial}{\partial x^i}$.

(iii) The isotropic metric space whose scalar curvature is not zero admits a *simple* group G_{10} of collineations, as already discussed[1] in connection with the groups of motions.

(iv) The space for which $P + Q + PQ = \text{const.} = \lambda$, $2P' - P^2 - 2P = \text{const.} = \mu$, (P and $Q \neq -1$) admits $G_7 = L_6 + N_1$ of collineations. The generator of N_1 is given by $(P + 1)\, x^i \dfrac{\partial}{\partial x^i}$. In particular case, when $\mu = 0$, we get projectively flat space ; $\lambda = 0$ in addition giving the ordinary flat space.

II. Groups of local applicabilities:

We shall now show that the solutions of the equations

$$S\, B^i_{jkl} \equiv u^r B^i_{jkl\,|\,r} + u^r_{\,|\,j}\, B^i_{rkl} + u^r_{\,|\,k}\, B^i_{jrl} + u^r_{\,|\,l}\, B^i_{jkr} - u^i_{\,|\,r}\, B^r_{jkl} = 0 \tag{10}$$

we shall calculate the Poisson brackets and see whether two, three or more (if possible) of its repeated alternations form a closed set. The differential equations obtained at the last stage of closure must be compatible for the groups of finite order. The compatibility conditions set up a relation between the constants of structure, which when solved determine the possible sub-groups of finite order. These, and only these, sub-groups will be called *Lie-groups* of local applicabilities for our spaces.

Sub-groups of order $2 - \mathrm{M}_2$:

The following symbols generate the M_2's of M_∞:

$$\mathrm{D}_1 = x^i \frac{\partial}{\partial x^i};\ \mathrm{D}_2 = \begin{cases} \log \mathrm{X}\, x^i \dfrac{\partial}{\partial x^i} \rightarrow \mathrm{C}^1_{12} = 2,\ \mathrm{C}^2_{12} = 0 \\ \mathrm{X}^l x^i \dfrac{\partial}{\partial x^i} \rightarrow \mathrm{C}^1_{12} = 0,\ \mathrm{C}^2_{12} = 2l \end{cases}$$

(l being arbitrary).

Sub-groups of order $3 - \mathrm{M}_3$:

The following symbols generate the M_3's of M_∞ :

$$\mathrm{D}_1 = x^i \frac{\partial}{\partial x^i};\ \mathrm{D}_2 = \begin{cases} \log \mathrm{X}\, x^i \dfrac{\partial}{\partial x^i};\ \mathrm{D}_3 = (\log \mathrm{X})^2 x^i \dfrac{\partial}{\partial x^i} \\ \mathrm{X}^l x^i \dfrac{\partial}{\partial x^i};\ \mathrm{D}_3 = \mathrm{X}^{-l} x^i \dfrac{\partial}{\partial x^i} \end{cases}$$

(l being arbitrary).

The basis of these M_3's can be chosen so as to have the following structure

$$(\mathrm{D}_1, \mathrm{D}_2) f = \mathrm{D}_1 f,\ (\mathrm{D}_1, \mathrm{D}_3) f = 2\mathrm{D}_2 f,\ (\mathrm{D}_2, \mathrm{D}_3) f = \mathrm{D}_3 f$$

so that these M_3's of local applicabilities are isomorphic to the projective group on the line, studied by Bianchi.[4]

I express my thanks to Prof. D. D. Kosambi for his kind suggestions and helpful discussions during the preparation of this paper.

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"Simply Transitive groups as groups of collineations in Path-Spaces."

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In the present note I prove the following theorem, which really amounts to a generalization of a similar theorem in Riemann spaces due to Bianchi.*

Any simply transitive group in n variables is the group of collineations (complete or partial) of an infinity of spaces V_n defined by "paths", i.e. the solution curve of

$$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0 \ ; \ \dot{x}^i = \frac{dx^i}{dt} \ etc \ , \ i = 1, \ldots, n \tag{1}$$

The intrinsic differential geometry of the path-spaces has already been developed by Kosambi✝ from two main assumptions: (a) the tensor invariance of all fundamental equations, including (1), and (b) the existence of a vectorial operator, D, the vanishing of which defines a parallelism and makes the solutions of (1) autoparallel lines.

Let $\underset{\sigma}{u}^i(x)$ for $i, \sigma = 1, \ldots, n$ be the components of the infinitesimal transformations of a simply transitive group. If we denote by u^σ_i the co-factor of $\underset{\sigma}{u}^i$ in the determinant $|\underset{\sigma}{u}^i|$ divided by this determinant, then

$$u^\sigma_j \underset{\sigma}{u}^i = \delta^i_j \ ; \ u^\sigma_j \underset{\tau}{u}^j = \delta^\sigma_\tau . \tag{2}$$

In order that the group may be a group of collineations in the path-spaces defined by (1) it is necessary and sufficient

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* L.P.Eisenhart: " Continuous groups of Transformations (pp. 218, 76).

✝ D.D.Kosambi: " Parallelism and Path-spaces" Math. Zeitschrift Bd. 37. 1933. pp. 608-618.

that the equations of variation of (1), namely

(3) $\ddot{u}^i_\sigma + \dot{u}^r_\sigma \alpha^i_{;r} + u^r_\sigma \alpha^i_{,r} = 0$

admit a set of solutions.(Comma and semicolon denote differentiation with respect to x^i and $\dot{x}^i$ respectively).

The equations (3) can be put in invariantive form

(3,a). $D^2 u^i_\sigma = P^i_r u^r_\sigma + D\varepsilon^i,$

where $$Du^i = \dot{u}^i + \tfrac{1}{2}\alpha^i_{;k} u^k + \varepsilon^i$$
$$\varepsilon^i = \alpha^i - \tfrac{1}{2}\dot{x}^k \alpha^i_{;k}$$

and P^i_r is the mixed tensor of the space.

Multiplying (3) by u^σ_k and summing for σ, we have

(4) $\alpha^i_{,k} - \alpha^i_{;j} \dot{x}^r L^j_{rk} + \alpha^r L^i_{rk} - L^i_{mk,r} \dot{x}^m \dot{x}^r + L^h_{rk} L^i_{mh} \dot{x}^m \dot{x}^r = 0,$

where the functions L^i_{jk} are defined by

(5) $L^i_{jk} = -u^\sigma_k u^i_{\sigma,j} = u^i_\sigma u^\sigma_{k,j}$

The conditions of compatibility of (4) are obtained, after eliminating $\alpha^i_{,k}$ by means of (4), in the form

(6) $\dot{x}^m \alpha^i_{;p} L^p_{mkr} = \alpha^m L^i_{mkr} + \dot{x}^m \dot{x}^p \Big\{ L^i_{prk,m} + L^i_{ph} L^h_{mkr} + L^h_{mk} L^i_{phr} + L^h_{mr} L^i_{pkh} \Big\},$

where $L^i_{jkl} = L^i_{jl,k} - L^i_{jk,l} + L^h_{jl} L^i_{hk} - L^h_{jk} L^i_{hl}$

By making use of the conditions

$$C^\alpha_{\beta\gamma} + C^\alpha_{\gamma\beta} = 0, \quad C^\alpha_{\beta\gamma} C^\sigma_{\alpha\tau} + C^\alpha_{\gamma\tau} C^\sigma_{\alpha\beta} + C^\alpha_{\tau\beta} C^\sigma_{\alpha\gamma} = 0,$$

which the constants of structure must satisfy, we have

$$L^i_{jkl} \stackrel{*}{=} 0$$

Therefore, the conditions (6) are identically satisfied for all values of α^i and consequently the system of equations (4) is completely integrable; hence the theorem.

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* D. D. Kosambi: "Parallelism and Path-spaces" Math. Zeitschrift Bd. 37. 1933. pp. 608-618.

* L.P. Eisenhart: "Continuous groups of Transformations" (pp. 218, 76).

T H I R D   T H E S I S

On the geometry of the spaces of Riemann constructed by representing hyperquadrics as points.

I N T R O D U C T I O N.

In the volumes LIV and LV of "Bulletin de la Société mathématique de France (1926-27)" Prof. E. Cartan has determined the symmetric real spaces of Riemann - the linear elements of these spaces being positive definite - by the method of rectangular <u>repère mobile</u>. These spaces are characterized by the property that the riemannian curvature of any fact whatsoever is conserved by the parallel transport. Analytically, this property is expressed by the fact that the tensor derivative of the Riemann-Christoffel tensor of these spaces is identically zero:

$$R_{ijkh/l} = 0$$

E. Cartan calls them the spaces $\mathcal{E}$. The space is called reducible if its linear element ds^2 is of the form

$$ds^2 = ds_1^2 + ds_2^2$$

ds_1^2 and ds_2^2 being the linear elements of n_1 and n_2 - $n_1 + n_2 = n$ - independent variables.

The spaces of Riemann of which ds_1^2 and ds_2^2 are the linear elements belong to the class of the spaces $\mathcal{E}$ under consideration. Hence the determination of the spaces $\mathcal{E}$ reduces to the determination of the irreducible spaces $\mathcal{E}$.

In section I of this work, I propose to expose, in brief, the methods of E. Cartan for the determination of the real irreducible spaces of Riemann $\mathcal{E}$ whose linear elements are positive definite.

E. Cartan has shown that out of the various classes the first class - class A, I - of these irreducible spaces is obtained by representing, by a point in a space of $N = \frac{1}{2} n(n+1) - 1$ dimensions, a definite positive or negative quadratic form; Two forms which differ only by a factor are represented by the same point or, in other words, the hyperquadrics in E_n are represented as geometric points in E_N.

These spaces must admit a real projective group of displacements G defined by the following table:

(G):

$X_{\alpha\beta} - X_{\beta\alpha}$
$i X_{\alpha\alpha}$
$i(X_{\alpha\beta} + X_{\beta\alpha})$

where

$$X_{\alpha\beta} = x_\alpha \frac{\partial}{\partial x_\beta},$$

and

$$(X_{\alpha\beta} X_{\beta\gamma}) = X_{\alpha\gamma}$$

$$(X_{\beta\alpha} X_{\gamma\beta}) = - X_{\gamma\alpha} .$$

I propose, in section II of this work, to indicate to the subject of this class of irreducible spaces

(1). a geometrical interpretation of their ds^2,

(2). a determination of their totally geodesic surfaces.

S E C T I O N I.

Symmetric spaces of Riemann.

1. From the definition of the spaces $\mathcal{E}$ it is evident that the group of holonomie, that is to say the group of rotations which the vectors issuing from a point A undergo when these vectors are transported by parallelism along any arbitrary cycle, conserves the riemannian curvature of the space at A and consequently leaves the form of Riemann

$$\sum_{i,j,k,h} R_{ijkh}\, x^i y^j x^k y^h$$

invariant. This form defines analytically the riemannian curvature of the space.

The first method of determination of the spaces $\mathcal{E}$ is based on the investigation of all the orthogonal groups which are susceptible of being the groups of holonomie of a space $\mathcal{E}$, that is to say which leave the form of Riemann invariant. This method rests on the properties of the real orthogonal groups.

Moreover, as we shall see, these spaces admit a transitive group (G) of displacements such that the sub-group, leaving a given point. A invariant, is the group of holonomie. Therefore, the second method of determination will consist of finding all the possible structures of the groups of displacements and trying to deduce from each of them the corresponding group of holonomie. The determination of the irreducible space $\mathcal{E}$, by this method, requires a knowledge of the results of a memoir " Les groupes réels simple, finis et continus" * in which E. Cartan has investigated all the real simple structures which correspond to the same type of complex structures.

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* Annales de L'École Norm sup. tome 31, 1914.

Statement of the problem in the field of the theory of groups.

2. Let us attach to a particular point A_0 of a real space of Riemann a rectangular repère (T_0) consisting of n unit orthogonal vectors. In the fascicule IX of "Mémorial des Sc.Math" E. Cartan has shown that the riemannian curvature of the space at the point A_0 is analytically defined by the form of Riemann

$$(1) \qquad R = \sum_{(ij),(kh)} R_{ij,kh}\, p_{ij}\, p_{kh},$$

where the quantities p_{ij} are the components of a bivector formed of any two vectors (x_h) and (y_h) issuing from A_0 such that

$$p_{ij} = x_i y_j - x_j y_i .$$

The riemannian curvature in the direction plane defined by the vectors (x_i) and (y_i) is given by

$$\sum_{(ij),(kh)} R_{ij,kh}\, p_{ij}\, p_{kh} \Big/ \sum_{(ij)} p_{ij}^2 ,$$

where the denominator is the measure of the corresponding bivector.

3. Transporting by parallelism the vectors issuing from the point A_0 along a cycle we find a rotation which these vectors undergo after the displacement; the rotations associated with different cycles issuing from A_0 generate a group

which is called the group of holonomie of the space. We shall call Γ the group of holonomie and denote its order by r. The variables which are transformed by Γ are the components (x_i) of an arbitrary vector.

At the point A_0 we shall consider not only the repère (T_0) but all those which are deduced from it by a rotation of the group of holonomie Γ. Attach to a point A the repères (T) which are deduced from (T_0) by the parallel transport of (T_0) along an arbitrary path joining A_0 and A. the repères (T) depend on r parameters. Also all the repères (T) at A are deduced from each other by the rotations of the group of holonomie.

The formulae of structure of the space are given by*

$$
(2) \qquad \begin{aligned}
\omega_i' &= \sum_k [\omega_k \omega_{ki}] \\
\omega_{ij}' &= \sum_k [\omega_{ik}\omega_{kj}] - \sum_{(kh)} R_{ij,kh} [\omega_k \omega_h],
\end{aligned}
$$

where $\omega_1, \ldots\ldots, \omega_n$ are linear with respect to the differentials of n coordinates which give the rectangular coordinates of the point A′ with respect to the repère (T) attached at an infinitely near point A; and $\omega_{ij} = -\omega_{ji}$ are linear with respect to the differentials of n coordinates and r parameters of the repères.

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* Mémorial des Sc. Math. Fas. IX.

4. We shall now determine the form of Riemann R from the knowledge of the group of <u>holonomie</u> Γ. Let the group Γ be generated by the r independent infinitesimal transformations

$$U_\alpha f = \sum_{(ij)} a_{\alpha ij} \left(x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right), \tag{3}$$

where we have

$$a_{\alpha ij} = - a_{\alpha ji}$$

The components ω_{ij} can be expressed by the form

$$\omega_{ij} = \sum_\rho a_{\rho ij}\, \theta_\rho , \tag{4}$$

for the infinitesimal rotation makes part of the group of holonomie; $\theta_1, \ldots\ldots, \theta_r$ are some suitable forms of Pfaff.

The n + r forms

$$\omega_1, \ldots, \omega_n; \theta_1, \ldots\ldots, \theta_r$$

are linearly independent. By substituting the values of ω_{ij} in the equations of structure we get

$$\omega_i' = \sum_{\rho, \kappa} a_{\rho \kappa i} \, [\omega_\kappa \theta_\rho]$$

$$\sum_\rho a_{\rho ij} \, \theta_\rho' = \sum_{(\lambda\mu), \kappa} \left(a_{\lambda i \kappa} a_{\mu \kappa j} - a_{\mu i \kappa} a_{\lambda \kappa j} \right) [\theta_\lambda \theta_\mu] - \sum_{(\kappa h)} R_{ij,\kappa h} \, [\omega_\kappa \omega_h] .$$

From the equations of structure of S.Lie

$$(U_\lambda U_\mu) f = \sum_\rho c_{\lambda\mu\rho} U_\rho f$$

we find

$$\boxed{\sum_\kappa \left(a_{\lambda i \kappa} a_{\mu \kappa j} - a_{\mu i \kappa} a_{\lambda \kappa j} \right) = \sum_\rho c_{\lambda\mu\rho} \, a_{\rho ij}} \tag{5}$$

Therefore, the second set of the equations of structure of the space $\mathcal{E}$ becomes

$$(6) \quad \sum_{\rho} a_{\rho ij} \left\{ \theta_\rho' - \sum_{(\lambda\mu)} c_{\lambda\mu\rho} [\theta_\lambda \theta_\mu] \right\} = -\sum_{(kh)} R_{ij,kh} [\omega_k \omega_h].$$

These equations can be regarded as the linear equations involving r unknowns.

$$\theta_\rho' - \sum_{(\lambda\mu)} c_{\lambda\mu\rho} [\theta_\lambda \theta_\mu].$$

There exist evidently r of these equations which are linearly independent. We have, therefore, a relation of the form

$$\theta_\alpha' - \sum_{(\lambda\mu)} c_{\lambda\mu\alpha} [\theta_\lambda \theta_\mu] = -\sum_{(kh)} b_{\alpha kh} [\omega_k \omega_h],$$

where $b_{\alpha kh} = -b_{\alpha hk}$ are constants.

We obtain, therefore,

$$R_{ij,kh} = \sum_{\rho} a_{\rho ij} b_{\rho kh}$$

whence

$$\frac{1}{2} \frac{\partial R}{\partial p_{kh}} = \sum_{(ij)} R_{ij,kh} p_{ij} = \sum_{\rho} b_{\rho kh} \left(\sum_{ij} a_{\rho ij} p_{ij} \right).$$

By associating to the infinitesimal transformations $U_\alpha f$ the forms

$$\varsigma_\alpha^\xi = \sum_{(ij)} a_{\alpha ij} p_{ij}$$

we see that the partial derivatives of R are expressed linearly by means of the r forms ς_α^ξ. Accordingly the form of Riemann is a quadratic form with respect to the r forms ξ_α associated to the r infinitesimal transformations of the group of <u>holonomie</u> Γ.

Now R can be written as

$$R = \sum_{\alpha,\beta} A_{\alpha\beta}\, \xi_\alpha \xi_\beta \quad (A_{\alpha\beta} = A_{\beta\alpha})$$

We have

$$\frac{1}{2}\frac{\partial R}{\partial p_{\kappa h}} = \sum_{\rho,\sigma} A_{\rho\sigma}\, a_{\rho\kappa h}\, \xi_\sigma .$$

Therefore,

$$b_{\alpha\kappa h} = \sum_{\rho} A_{\alpha\rho}\, a_{\rho\kappa h}$$

The forms ω_i and θ_α, now, satisfy the following relations:

$$(7)\quad \left\{\begin{aligned} \omega_i' &= \sum_{\rho,\kappa} a_{\rho\kappa i}\,[\omega_\kappa \omega_\rho] \\ \theta_\alpha' &= \sum_{(\lambda\mu)} C_{\lambda\mu\alpha}\,[\theta_\lambda \theta_\mu] - \sum_{\rho,(\kappa h)} A_{\alpha\rho}\, a_{\rho\kappa h}\,[\omega_\kappa \omega_h]. \end{aligned}\right.$$

5. We shall, now, obtain some relations which the constants $a_{\rho\kappa i}, C_{\lambda\mu\alpha}, A_{\alpha\beta}$ must satisfy in order that

(i) the form of Riemann $\sum_{\alpha,\beta} A_{\alpha\beta}\,\xi_\alpha\xi_\beta$ be invariant by the group of <u>holonomie</u> Γ. This gives the following conditions

$$(8)\quad \boxed{\sum_{\rho}\left(C_{\alpha\lambda\rho}A_{\mu\rho} + C_{\alpha\mu\rho}A_{\lambda\rho}\right) = 0 \quad (\alpha,\lambda,\mu = 1,\dots,r)},$$

(ii) the form of Riemann must satisfy the identities of Ricci. For that we get

$$(9)\quad \boxed{\sum_{\rho,\sigma} A_{\rho\sigma}\left(a_{\sigma ij}\,a_{\rho\kappa h} + a_{\sigma j\kappa}\,a_{\rho ih} + a_{\sigma\kappa i}\,a_{\rho jh}\right) = 0}.$$

6. Let us suppose that we have a system of constants $a_{\alpha ij}$, $C_{\alpha\beta\gamma}$, $A_{\alpha\beta}=A_{\beta\alpha}$ satisfying the relations (5), (8) and (9). We shall prove that there exists a space of Riemann $\mathcal{E}$ admitting Γ for the group of holonomie and R for the form of Riemann; moreover this space possesses the property that its curvature is conserved by the parallel transport.

There exists a group whose constants of structure enter in the equations of structure (7) of the space. The brackets of the infinitesimal transformations $X_i f$ and $Y_\alpha f$ of this group are given by

$$(10)\qquad \boxed{\begin{aligned} (X_i X_j)f &= -\sum_{\rho,\sigma} a_{\rho ij} A_{\rho\sigma} Y_\sigma f, \\ (X_i Y_\alpha)f &= \sum_k a_{\alpha ik} X_k f, \\ (Y_\alpha Y_\beta)f &= \sum_\rho C_{\alpha\beta\rho} Y_\rho f. \end{aligned}}$$

It can be shown easily that the identities of Jacobi are satisfied in virtue of the relations given by (5), (8) and (9).

It is, therefore, possible to find $n+r$ expressions of Pfaff ω_i, θ_α of $n+r$ independent variables satisfying the equations of structure (7).

Consider n equations of Pfaff

$$\omega_1 = \omega_2 = \cdots\cdots = \omega_n = 0;$$

they are completely integrable because ω_i' are all zero in virtue of the equations (7). Let

$$u_1, \ldots\ldots, u_n$$

be a system of independent integrals of these equations and let $\nu_1, \nu_2, \ldots \ldots \nu_r$ be a system of other independent functions which are also independent of u_i. If we take u_i and ν_α as the independent variables, the ω_i become linear in $du_1, \ldots, du_n$ and the quadratic form

$$(11) \qquad ds^2 = \omega_1^2 + \cdots + \omega_n^2 = \sum_{i,j} g_{ij}\, du_i\, du_j$$

can be regarded as the ds^2 of a space of Riemann.

By making use of the first equations of structure of the space

$$\delta\omega_i(d) - d\omega_i(\delta) = \sum_{\rho,\kappa} a_{\rho\kappa i}\left[\omega_\kappa(\delta)\theta_\rho(d) - \omega_\kappa(d)\theta_\rho(\delta)\right]$$

and denoting by d the differentiation with respect to one of the variables u_i and by δ the differentiation with respect to any one of the variables we find that g_{ij} is independent of ν_α.

The space Riemann whose ds^2 is given by (11) is referred to a rectangular <u>repère mobile</u> depending on r parameters ν_α. The tensor of Riemann-Christoffel is given by

$$R_{ij,\kappa h} = \sum_{\rho,\sigma} A_{\rho\sigma}\, a_{\rho ij}\, a_{\sigma\kappa h}.$$

whence

$$R = \sum A_{\rho\sigma}\, \xi^\xi_\rho\, \xi^\xi_\sigma .$$

We are going to demonstrate that Γ is the group of holonomie of the space and the riemannian curvature is conserved by the parallel transport.

The infinitesimal transformations of the group of holonomie which are associated with an elementary cycle are given by

$$V_{kh} f = \sum_{(ij)} R_{ij,kh} \left(x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right) = \sum_{\rho,\sigma} A_{\rho\sigma} a_{\sigma kh} U_\rho f .$$

Therefore, these transformations belong to the group Γ. We prove further that they generate an invariant sub-group of Γ by establishing the following equations by the help of the equations (8) and (5)

$$(U_\alpha V_{kh}) f = \sum_i \left(a_{\alpha ik} V_{ih} f - a_{\alpha ih} V_{ik} f \right)$$

Lastly in order to prove that the riemannian curvature is conserved by the parallel transport we consider two repères (T) and (T') attached to the infinitely near points (A) and (A') respectively.

The form of Riemann has the same coefficients $R_{ij,kh}$, at these two points, referred to these ~~two~~ repères. Let (T_1) be the repère of origin A' obtained by transporting by parallelism ~~that~~ the repère (T) from A to A'. We obtain (T') from (T_1) by the rotation of components ω_{ij}, that is to say by a rotation of the group of holonomie Γ; this group leaves the form of

Riemann invariant. Therefore, the coefficients of the form R are the same referred to either (T) or (T_1); accordingly the transport by parallelism conserves the curvature.

The method based on the group of holonomie.

7. Prof. E. Cartan has shown that*

" The necessary and sufficient condition in order that a space $\mathcal{E}$ be irreducible is that its group of holonomie does not leave any multiplicity real plane invariant."

We shall show, now, how we can determine the real orthogonal groups Γ which do not leave any multiplicity real plane invariant.

Let us investigate whether the group Γ can leave a multiplicity imaginary plane P invariant; if so it will also leave the conjugate imaginary plane P_0 invariant. E. Cartan has proved that " the only multiplicity imaginary planes which can be invariants by the group Γ are the totally isotrope multiplicities of n/2 dimensions. The group Γ can, then, be considered as a group of real parameter and n/2 complex variables which leaves a Hermitian positive definite form invariant."

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* Bull. Soc. Math. de. France tome 54, 1926.

8. We obtain some informations about the structure of the real orthogonal groups by putting the bases of their infinitesimal transformations under a normal form. Let us take real rectangular coordinates x_i. Any orthogonal infinitesimal transformation.

$$Uf = \sum_{(ij)} a_{ij}\left(x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i}\right)$$

is defined by a system of bivectors $a_{ij} = -a_{ji}$. We shall say that the square of the measure of this system of bivectors is the Scalar square of the infinitesimal transformation, and we shall write

$$U/U = \sum_{(ij)} a_{ij}^2$$

Similarly the Scalar product of the two transformations Uf, Vf corresponding to the system of bivectors a_{ij} and b_{ij} is given by the relation

$$U/V = \sum_{(ij)} a_{ij}\ b_{ij}$$

The real orthogonal group Γ of order r will have a normal base if the r infinitesimal transformations $U_1f, \ldots\ldots, U_rf$ have their Scalar squares equal to unity and if the Scalar product of any two of them is zero. It is always possible to reduce Γ to the normal base. We can, therefore, suppose that

$$\sum_{(ij)} a_{\alpha ij}^2 = 1$$

$$\sum_{(ij)} a_{\alpha ij}\, a_{\beta ij} = 0 \quad (\alpha \neq \beta).$$

We, then, obtain a remarkable property of the constants of structure of the group by using the equations (5)

$$C_{\alpha\beta\gamma} = C_{\beta\gamma\alpha} = C_{\gamma\alpha\beta} = -C_{\beta\alpha\gamma} = -C_{\gamma\beta\alpha} = -C_{\alpha\gamma\beta}\,.$$

Let us consider, now, the group Γ' which indicates how Γ transforms the forms ξ_α associated to the transformations $U_\alpha f$. We have

$$U_\alpha(\xi_\beta) = \sum_\rho C_{\alpha\beta\rho}\,\xi_\rho\,,$$

whence,

$$U'_\alpha f = \sum_{(ij)} C_{\alpha ij}\left(\xi_j \frac{\partial f}{\partial \xi_i} - \xi_i \frac{\partial f}{\partial \xi_j}\right);$$

and we see that Γ' is also a real orthogonal group. This, in the theory of groups, is called the adjoint group of the given group.

There corresponds an invariant sub-group of Γ, to any multiplicity plane which is invariant by Γ'. If the group Γ' transforms a certain number of linear forms in $\xi_1, \ldots\ldots, \xi_r$, for example

$$\begin{array}{l} a_1\xi_1 + a_2\xi_2 + \cdots + a_r\xi_r\,, \\ b_1\xi_1 + b_2\xi_2 + \cdots + b_r\xi_r\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots; \end{array}$$

this means that the brackets of any transformation of Γ with the different transformations

$$\begin{array}{l} a_1U_1f \quad \ldots\ldots \quad a_rU_rf, \\ b_1U_1f \quad \ldots\ldots \quad b_rU_rf, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

are the linear combinations of these last transformations.

Thus the investigation of the invariant sub-groups of Γ reduces to that of the multiplicity planes invariant by Γ'. If we confine ourselves to those which are real, we see that we can suppose ξ^{ρ} to be chosen in such a manner as to divide them in a certain number of series

$$\xi_1, \ldots, \xi_\rho;$$
$$\xi_{\rho+1}, \ldots, \xi_t;$$
$$\ldots\ldots$$

possessing the following properties:

1. The variables of each series are transformed into themselves by the group Γ';
2. It is impossible to decompose any series into the partial series in each of which the variables are transformed into themselves.

It follows, therefore, that the only constants of structure $c_{\alpha\beta\gamma}$ which can be different from zero are those for which the three indices belong to the same series. In other words, the group Γ is decomposable in a certain number of invariant real sub-groups exchangeable among them. These sub-groups do not contain any invariant sub-group; that is to say they are simple groups.*

E. Cartan has proved that if an orthogonal real group does not admit any invariant real sub-group, it does not admit any imaginary sub-group also.

We thus arrive to a final theorem due to E. Cartan.

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* Bull. Soc. Math. tome 54, 1926.

"Any orthogonal real group is decomposable in a certain number of simple invariant real sub-groups exchangeable among them."

9. In the theory of simple groups, there exists a quadratic form playing an important rôle. This is the form $\varphi(e)$ which gives the sum of the squares of roots of the characteristic equations relative to the arbitrary infinitesimal transformation $\sum_i e_i U_i f$ of the group

$$\boxed{\varphi(e) = \sum_{i,j,\rho,\sigma} e_i e_j C_{i\rho\sigma} C_{j\sigma\rho}}$$

Replacing the variables e_i by the variables ξ_i, we see that the form $\varphi(\xi)$ thus obtained is invariant by the group Γ', which is the adjoint group of Γ. If Γ is simple, the only quadratic form invariant by the orthogonal group Γ' is, to a constant factor, $\sum_i \xi_i^2$ we have, therefore

$$\sum_{(\rho\sigma)} C_{\alpha\rho\sigma} C_{\beta\rho\sigma} = 0 \quad (\alpha \neq \beta) \; ; \sum_{(\rho\sigma)} C_{\alpha\rho\sigma}^2 = H$$

Therefore

$$\varphi(e) = -2H(e_1^2 + \cdots + e_r^2).$$

The simple groups into which the group Γ is decomposed possess the property that their form $\varphi(e)$ is definite negative. We say that they are groups of <u>unitary</u> simple real structures.

10. Now we indicate a method of determination of all the real orthogonal groups which do not leave any multiplicity real plane invariant.

They are divided into two classes:

(1). those which do not leave any multiplicity imaginary plane invariant ;

(2). those which leave totally isotrope imaginary plane invariant.

The determination of the groups of the second class is very easy. We determine all the linear groups (a) whose parameters are real and the variables are complex (b) which do not leave any multiplicity plane invariant (c) which can be decomposed into simple unitary real sub-groups and (d) which do not admit any antiinvolution of the first kind.*

For the determination of the group of the first class we start from simple unitary groups whose parameters are real and variables are complex (a) which do not leave any multiplicity plane invariant and (b) which are their proper corelatives. Each simple group has an index (± 1) which indicates the kind of antiinvolution that the group admit. It is necessary that the product of all the indices be equal to unity.

11. From the previous considerations of the structure of the real orthogonal groups, E. Cartan has arrived to the following conclusions:

The groups of holonomie which are to be determined belong to three different types:

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* Jour. Math. tome X. 1914. 156-162.

(1). The group Γ does not leave any multiplicity imaginary plane invariant; this is simple group;

(2). The group Γ does not leave any multiplicity imaginary plane invariant; this is semi-simple and is obtained by the multiplication of a simple group of three parameters and two variables by a simple group of 2p variables leaving an exterior quadratic form and a positive definite Hermitian form invariants;

(3). The group Γ leaves a totally isotrope multiplicity imaginary plane invariant. This group is obtained by the multiplication of a simple unitary group and a group of one parameter $iu\frac{\partial f}{\partial u}$.

12. Here are some important results due to E. Cartan:

To any given group of holonomie there corresponds a class of the spaces of Riemann for which the form of Riemann R is determined to a constant factor and preserves, at any point of the space and for any direction plane, a constant sign.

We can, therefore, call - in each class - some irreducible spaces of positive curvature and some irreducible spaces of negative curvature. The curvature is defined by one parameter alone. This property gives a generalization of the space of constant riemannian curvature.

Another important property is obtained by the consideration of the contracted curvature tensor. The group of <u>holonomie</u>, leaving the form of Riemann invariant, leaves also the contracted form

$$\sum_{i,j} R_{ij}\, x_i x_j = \sum_{i,j,k} R_{ik,jk}\, x_i x_j$$

invariant.

Now the group Γ, not leaving any multiplicity real plane invariant, can leave only a real quadratic form invariant. Accordingly the contracted form is proportional to the fundamental form

$$\sum_i x_i^2$$

It results from this that the irreducible spaces $\mathcal{E}$ are spaces of constant curvature of the second kind (Mémorial. Sc. Math. Fas. IX n^0 36).

Returning to the expressions of $R_{ik,jk}$, we find that

$$\sum_{\rho,k} A_\rho a_{\rho ik} a_{\rho jk} = 0 \ (i \neq j),$$

$$\sum_{\rho,k} A_\rho (a_{\rho ik})^2 = C \quad ,$$

the Constant C is positive for the spaces of positive curvature, and negative for the spaces of negative curvature.

We shall, now, demonstrate that Γ is the greatest continuous group of rotations possessing the property that it leaves the form of Riemann invariant.

Suppose there exists an infinitesimal rotation Vf which is not a part of Γ and which leaves the form of Riemann R invariant. As R is a non-degenerate quadratic form of the forms ξ_α associated to the infinitesimal transformations of Γ, it is necessary that the transformation Vf makes ξ_α to undergo a linear substitution; this means that the group Γ is invariant by the transformation of Vf. The transformations Vf and $U_\alpha f$ form an orthogonal real group and accordingly we see that

$$(VU_\alpha) \equiv 0$$

The new group, which is real and orthogonal, admits a one-parameter invariant sub-group. This is possible only if it leaves a multiplicity imaginary plane invariant, but then the group would already admit a one-parameter invariant sub-group, and a real orthogonal group, which does not leave any multiplicity real plane invariant, cannot admit any one-parameter invariant sub-group. Thus we arrive at a contradiction.

<u>Method based on the group of displacements.</u>

13. First of all we shall prove that the space whose linear element is given by

$$ds^2 = \omega_1^2 + \cdots + \omega_n^2 = \sum_{i,j} g_{ij}\, du_i\, du_j$$

admits a group of rigid displacements having the following structure:

$$\text{(G)} \quad \text{(10)} \quad \boxed{\begin{aligned} (X_i X_j) &= -\sum_{\rho,\sigma} a_{\rho ij} A_{\rho\sigma} Y_\sigma f, \\ (X_i Y_\alpha) &= \sum_{k} a_{\alpha ik} X_k f, \\ (Y_\alpha Y_\beta) &= \sum_{\rho} c_{\alpha\beta\rho} Y_\rho f. \end{aligned}}$$

For demonstrating the theorem we consider n+ r forms of Pfaff ω_i and θ_α in $u_1, \ldots\ldots, u_n$; $\nu_1, \ldots\ldots, \nu_r$ and the equations

$$(12) \quad \begin{cases} \omega_i(u', \nu'; du') = \omega_i(u, \nu; du) \\ \theta_\alpha(u', \nu'; du', d\nu') = \theta_\alpha(u, \nu; du, d\nu) \end{cases}$$

They form a completely integrable system of total differential equations. Here u_i' and ν_α' are the unknown functions of u_i and ν_α. The covariant bilinears of two members of any of the above equations are evidently equal. This can be seen by taking into consideration the equations themselves and the constants which enter in the equations of structure (7).

The du_i' being linear with respect to du_i only, any solution of (12) is evidently of the form

$$u_i' = f_i(u)$$

$$\nu_\alpha' = \varphi_\alpha(u; \nu)$$

They define a point transformation of the space into itself and by this transformation we get

$$\sum_i [\omega_i(u', \nu'; du')]^2 = \sum_i [\omega_i(u, \nu; du)]^2$$

The ds^2 is, therefore, conserved and we get ∞^{n+r} isometric transformations of the space into itself. These displacements form a group (G) of which the equations of definition are given by (7) and the equations of structure are given by (10).

14. Let us prove that the space always admits a symmetry with respect to any of its points.

For that it is sufficient to consider the system of Pfaff:

$$(13)\quad \begin{cases} \omega_i(u',\nu';du') = -\omega_i(u,\nu;du) \\ \theta_\alpha(u',\nu';du',d\nu') = \theta_\alpha(u,\nu;du,d\nu) \end{cases}$$

The equations (13) are completely integrable as can be seen from the equations (7). We, therefore, deduce from them a second family of isometric transformations of the space. Consider the solution of this system of equations such that there correspond to some given numerical values of u_i and ν_α the same numerical values of u_i' and ν_α'. The isometric transformation defined by this solution leaves a point A and a repère (T) attached to this point invariant, but the components of a vector issuing from A change their signs. Therefore, any point M is transformed into the other point M′ situated on the geodesic AM and on the other side of A such that

$$AM' = AM.$$

Thus we get a symmetry with respect to a point A. This property is a characteristic property of the spaces $\mathcal{E}$. We shall, now, demonstrate a theorem:

" If a space of Riemann is such that the symmetry with respect to any of its points be an isometric transformation, the curvature of this space is conserved by the parallel transport."

We need the support of a remarkable construction of parallel transport due to E. Cartan for the demonstration of the above theorem.

For transporting by parallelism a direction from A to an infinitely near point A', it is sufficient to construct the geodesic AA' and to take the direction issuing from A' symmetric to the given direction with respect to the middle point C of AA'. Let us consider, now, an element plane defined by two given directions at A. The transport by parallelism of the element plane from A to A' will give the same direction as the symmetry with respect to C. Now this symmetry conserves the riemannian curvature of the space.

15. Before going to expose the second method of determination of the irreducible spaces $\mathcal{E}$, we shall point out a remarkable class of these spaces for which the group of holonomie is the adjoint group of a simple group, the latter being also unitary.

If we consider a simple unitary group for which $\varphi(e)$ has been reduced to the sum os squares, we have the following relations

$$(14) \qquad C_{\alpha\beta\gamma} = C_{\beta\gamma\alpha} = C_{\gamma\alpha\beta} = -C_{\beta\alpha\gamma} = -C_{\gamma\beta\alpha} = -C_{\alpha\gamma\beta}.$$

The adjoint group is generated by the transformations

$$U_\alpha f = \sum_{(ij)} C_{\alpha j i}\left(x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i}\right),$$

and the form of Riemann, if it exists, is

$$R = A \sum_{\rho} \Big[\sum_{(ij)} C_{\rho ij}\, p_{ij}\Big]^2.$$

The relations (9) become here

$$\sum_\rho \left(C_{\rho ij} C_{\rho kh} + C_{\rho jk} C_{\rho ih} + C_{\rho ki} C_{\rho jh}\right) = 0 .$$

Now they are verified. By taking into consideration the relations (14) they give analytically the identities of Jacobi

$$((X_i X_j) X_k) + ((X_j X_k) X_i) + ((X_k X_i) X_j) = 0 .$$

Therefore, there corresponds to each type of the simple unitary group a class of the irreducible spaces of Riemann .

16. We shall show that those of these spaces which have positive curvature are the representative spaces of the transformations of the simple unitary group which correspond to them.

The formulae of the structure of the space take the following forms here

$$(7) \quad \begin{cases} \omega_i' = \sum\limits_{k,\rho} C_{ik\rho} [\omega_k \theta_\rho], \\ \theta_\alpha' = \sum\limits_{(\lambda\mu)} C_{\lambda\mu\alpha} [\theta_\lambda \theta_\mu] + A \sum\limits_{(kh)} C_{\alpha kh} [\omega_k \omega_h]. \end{cases}$$

Consider the equations of Pfaff

$$\theta_i = m\omega_i \quad (i = 1, \ldots, r);$$

they give by exterior derivation

$$(m^2 - A) \sum_{(kh)} C_{ikh} [\omega_k \omega_h] = 0$$

Accordingly, if $m = \pm\sqrt{A}$, they are completely integrable.

If we take a solution of this, we are allowed to fix at each point of the space a particular rectangular repère (T). But we have then

$$\omega_i' = 2m \sum_{(kh)} C_{ikh} [\omega_k \omega_h] ;$$

the forms of Pfaff ω_i then define the <u>unitary</u> group under consideration. These are the components of the infinitesimal transformations $T_{\xi+d\xi}\,T_{\xi}^{-1}$ of the group, where ξ denote the parameters. The space of Riemann is, then, the representative space of the group of transformations.

It is also to be remarked that if the space has negative curvature, this sort of interpretation is not possible.

17. The function $\varphi(e)$, giving the square of the roots of the characteristic equation relative to an arbitrary infinitesimal transformation $\sum_i e_i X_i f$ of the group, has the form

$$\varphi(e) = \sum_{i,j,\rho,\sigma} e_i e_j C_{i\rho\sigma} C_{j\sigma\rho} .$$

In his thesis E. Cartan has given a theorem which is also true for the form $\varphi(e)$. The theorem for the form $\varphi(e)$ is enunciated as follows:

" The necessary and sufficient condition in order that a group be simple or semi-simple is that the form $\varphi(e)$ has its discriminant different from zero."

We shall, now, make the calculation for the group (G) of displacements whose structure is given by (10)

Denoting an arbitrary transformation of the group by

$$\sum_i e_i X_i f + \sum_\alpha \eta_\alpha Y_\alpha f$$

we write

$$-\frac{1}{2}\varphi(e,\eta) = \sum_{i,j} G_{ij}\, e_i e_j + 2\sum_{i,\alpha} G_{i\alpha}\, e_i \eta_\alpha + \sum_{\alpha,\beta} G_{\alpha\beta}\, \eta_\alpha \eta_\beta .$$

By simple calculation we get

$$G_{ij}=\sum_{\rho,\sigma,\kappa} A_{\rho\sigma}\, a_{\rho i\kappa}\, a_{\sigma j\kappa}\,,$$
$$G_{i\alpha}=0,$$
$$G_{\alpha\beta}=\sum_{(ij)} a_{\alpha ij}\, a_{\beta ij}-\frac{1}{2}\sum_{\rho,\sigma} C_{\alpha\rho\sigma}\, C_{\beta\sigma\rho}\,.$$

Taking into consideration the relations

$$\sum_{\rho,\kappa} A_\rho\, a_{\rho i\kappa}\, a_{\rho j\kappa}=0 \quad i\neq j$$
$$\sum_{\rho,\kappa} A_\rho\,(a_{\rho i\kappa})^2=C\,,$$

and supposing the group Γ decomposed into its simple sub-groups which are reduced to their normal forms, we have

$$G_{ii}=C\,,\quad G_{ij}=0\ (i\neq j)\,,$$
$$G_{i\alpha}=0\,,$$
$$G_{\alpha\alpha}=1+\sum_{(\rho\sigma)}(C_{\alpha\rho\sigma})^2=1+H_\alpha^2\,,\quad G_{\alpha\beta}=0\ (\alpha\neq\beta)\,.$$

Thus we obtain

$$-\frac{1}{2}\varphi(e,\eta)=C(e_1^2+\cdots+e_n^2)+\sum_\alpha(1+H_\alpha^2)\,\eta_\alpha^2$$

The part $\sum_\alpha(1+H_\alpha^2)\eta_\alpha^2$ of $-\frac{1}{2}\varphi(e,\eta)$ is positive definite whereas the part $C(e_1,\dots\, e_n)$ is positive or negative definite unless C be zero. We prove, now, that C is never zero.

The relations (10) show that the group G is its proper derived group; accordingly the equations

$$\frac{\partial\varphi}{\partial e_i}=\frac{\partial\varphi}{\partial\eta_\alpha}=0$$

define the greatest integrable invariant sub-group of G. If C were zero, this sub-group would be generated by the transformations $X_i f$. But we see from (10) that they do not generate any group.

18. We shall, now, discuss in what case the group G is not simple. E.Cartan[*] has shown that the sub-group g of the given group G, when generated as follows, coincide with the group G itself:

(1). when g is generated by the transformations $X_i f$,

(2). when g contains one or several linear combinations of the transformations $X_i f$,

(3). when g contains a transformation $Y_\alpha f$.

There remains the only possible case when the group g would be generated by a certain number s of transformations of the form

$$X_1 f + Z_1 f, \ldots\ldots\ldots, X_s f + Z_s f$$

the $Z_i f$ being the independent linear combinations of $Y_\alpha f$.

We deduce immediately that r+ s transformations

$$X_1 f, \ldots\ldots, X_s f \ ; \ Z_1 f, \ldots\ldots, Z_s f$$

would also form an invariant sub-group. But this is possible only if

$$\omega = n = r$$

The $Z_i f$ are not necessarily the real combinations of $Y_i f$. We have now some relations of the following forms:

$$(X_i + Z_i, X_j) = \sum_K A_{ijK}(X_K f + Z_K f),$$

$$(X_i + Z_i, Z_j) = \sum_K B_{ijK}(X_K f + Z_K f),$$

* Bull.Soc.Math. tome 55. 1927.

whence, according to the equations (10),

$$(15) \qquad (X_i X_j) = \sum_k A_{ijk} Z_k f, \quad (Z_i X_j) = \sum_k A_{ijk} X_k f,$$

$$(X_i Z_j) = \sum_k B_{ijk} X_k f, \quad (Z_i Z_j) = \sum_k B_{ijk} Z_k f$$

we have, therefore

$$(16) \qquad A_{ijk} = -A_{jik}, \quad B_{ijk} = A_{ijk}$$

The transformations $X_i f - Z_i f$ also form an invariant sub-group, because we have

$$(X_i - Z_i, X_j) = -\sum_k A_{ijk}(X_k f - Z_k f)$$

$$(X_i - Z_i, Z_j) = \sum_k A_{ijk}(X_k f - Z_k f)$$

The semi-simple group G is, therefore, decomposed into two simple sub-groups.

Now let

$$-\frac{1}{2}\varphi(e,\zeta) = C(e_1^2 + \cdots + e_n^2) + \sum_{\alpha,\beta} H_{\alpha\beta}\zeta_\alpha\zeta_\beta.$$

be the form which gives the semi-sum, the sign being changed, of the squares of the roots of the characteristic equations of the transformation

$$\sum (e_i X_i f + \zeta_\alpha Z_\alpha f).$$

The first invariant sub-group is defined by

$$e_i = \zeta_i$$

the second invariant sub-group will be defined by

$$\frac{\partial\varphi}{\partial e_i} + \frac{\partial\varphi}{\partial\zeta_i} = 0$$

or $$C e_i + \sum_k H_{ik}\zeta_k = 0.$$

These equations must be reduced to

$$\zeta_i + e_i = 0$$

Therefore,

$$H_{ij} = 0, \quad H_{ii} = C$$

$$-\frac{1}{2}\varphi(e,\varsigma) = C\,(e_1^2 + \cdots + e_n^2 + \varsigma_1^2 + \cdots + \varsigma_1^2)$$

The form obtained for $\varphi(e)$ shows that the transformations $Z_i f$ can be regarded as proportional to $Y_i f$ - the factor of proportionality being real if $C > 0$, purely imaginary if $C < 0$.

Stating now

$$Z_i f = m\,Y_i f$$

we find in virtue of (15) and (16)

$$A_{ijk} = m^2 C_{ijk}$$

$$(X_i Y_\alpha) = m \sum_k C_{i\alpha k} X_k f,$$

whence

$$a_{\alpha ij} = -m\,C_{\alpha ij}$$

The group is, therefore, the adjoint group of a simple <u>unitary</u> group and thus we come back to the class of spaces studied in paragraph (15) and (16).

We thus arrive at the following conclusion :

The group G of displacements of an irreducible space $\mathcal{E}$ is simple. The exceptional case aries when the class of spaces belong to the spaces of simple groups; for this class of spaces the group ($\mathcal{G}$) of displacements is decomposed into two isomorphic simple groups - these groups being real if the space is of positive curvature, conjugate imaginary if the space is of negative curvature.

19. We shall reduce the problem to that of the theory of real simple groups. Supposing that the group G is simple, we can formulate in a different manner the problem of the investigation of the irreducible spaces $\mathcal{E}$. The n infinitesimal transformations $X_i f$ are such that

$$(X_i(X_j\ X_k))$$

is deduced linearly from $X_k f$; also they do not belong to any sub-group of G.

Conversely, let us suppose we have fund out, in a simple group of n r parameters, n independent transformations $X_1 f, \ldots, X_n f$ such that

$$(X_i(X_j\ X_k)) \quad (i,j,k = 1,\ldots,n)$$

depend linearly on $X_i f$ alone, and such that $X_i f$ do not belong to any sub-group of G. The brackets $(X_i X_j)$ must give rise to r new independent infinitesimal transformations, because if this be not so, the transformations $X_i f$ and $(X_i X_j)$ would generate a sub-group of G. E. Cartan has shown* that the brackets $(X_i X_j)$ provide r new independent transformations which taken together generate a group having the same structure as the group Γ. It is necessary to add the supplementary condition that the group Γ is decomposed into the unitary groups.

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* Bull. Soc. Math. tome 55. 1927.

In " Journal de Mathematiques tome 6.1927" E. Cartan has shown that the investigation of the aggregate of the transformations $X_i f$ satisfying the above properties is reduced to the problem of finding the totally geodesic varieties, in the (Riemannian) group space of G, which are not the group spaces of a sub-group and which are contained in no representative varieties of any sub-group. These totally geodesic varieties are themselves the spaces of Riemann which are applicable on the spaces under investigation.

Thus the proposed problem is reduced to the investigation of the totally geodesic varieties of the representative spaces of the simple groups.

20. E. Cartan has stated the same problem from another point of view which leads it to a problem of the theory of group already being discussed by the same author.

Consider an irreducible space $\mathcal{E}$ of which the group of displacements G is simple. The group of <u>holonomie</u> Γ being given, let us change the signs of the coefficients $A_{\alpha\beta}$ of the form of Riemann.

$$R = \sum_{\alpha,\beta} A_{\alpha\beta}\,\xi_\alpha \xi_\beta .$$

Thus we get another irreducible space $\mathcal{E}'$ corresponding to another group of displacements (G'). Denoting by $X_i' f$, $Y_\alpha' f$ the real infinitesimal transformations of the group G', we see immediately that the isomorphism between the two groups G and G' is obtained by putting

$$X'_k f = iX_k f \quad (k = 1,\dots\dots,n)$$

$$Y'_\alpha f = Y_\alpha f \quad (\ = 1,\dots\dots,r);$$

but there does not exist any <u>real</u> isomorphic correspondence between the two groups, because the two forms $\varphi(e)$ are of different nature - one being negative definite and the other indefinite whose r squares are negative and n squares are positive

Thus we arrive at the following conclusion:

There exist two distinct real simple structures belonging to the same complex type for any irreducible space $\mathcal{E}$. There exists between the two real structures an isomorphic imaginary correspondence in which r real infinitesimal transformations of one of these groups correspond to r real transformations of the other and n real transformations of one of the groups correspond to n purely imaginary transformations of the other. Finally the form $\varphi(e)$ corresponding to one of the real structures is negative definite.

21. In the terminology of E. Cartan a <u>normal</u> isomorphic correspondence is defined in the following manner.

Let G and G' be two groups having real parameters and having two real simple structures (given before hand) belonging to the same complex type; let one of these groups be unitary. If the basis of the real infinitesimal transformations of G could be chosen in such a way that each of them corresponds to a real or imaginary infinitesimal transformations of G', the correspondence thus obtained between the two groups G and G' is called a normal isomorphic correspondence.

E. Cartan has shown in a memoir " Annales de l'Ecole Normale Sup. tome 31. 1914" that

(1). being given two distinct real simple structures corresponding to the same type of complex structure - one of these structures being unitary - we can establish a normal isomorphic correspondence between the two structures.

(2). the different normal isomorphic correspondences, that we can establish between the two structures, lead to the same group of holonomie Γ.

22. Discussing different types of real simple structures and indicating for each of them the nature of the group of holonomie Γ, E. Cartan has classified the irreducible spaces $\mathcal{E}$ into many types. I shall give only the table of the group corresponding to his class of spaces of type (A.I) because of our interest to discuss the geometry of this class of spaces in Section II of this work.

The transformations of the <u>unitary</u> group G' are defined by the relations*

$$X_{\alpha\beta} = -\bar{X}_{\beta\alpha} \quad (\alpha, \beta = 1, \ldots, l+1).$$

This group is isomorphic to the 'unimodular' group of a form of Hermite containing l+1 variables.

The group G is defined by the real transformations $X_{\alpha\beta}$. We obtain the following table corresponding to the irreducible space of type (A.I).

(A.I)	$X_{\alpha\beta} - X_{\beta\alpha}$
(G)	$i\,X_{\alpha\alpha}$ $i(X_{\alpha\beta} + X_{\beta\alpha})$

where $X_{\alpha\beta} - X_{\beta\alpha}$ indicate the real transformations of the <u>unitary</u> group G' which corresponds to the real transformations of the non-unitary group G; and

$$i\,X_{\alpha\alpha}\ ;\ i\,(X_{\alpha\beta} + X_{\beta\alpha})$$

correspond to purely imaginary transformations.

The difference between the number of positive and negative squares of the form $\varphi(e)$ corresponding to the group G is, in this case, l; the number of parameters of the group of <u>holonomie</u> Γ is $\frac{1}{2}l(l+1)$ and the number of variables of is $\frac{1}{2}\,l(l+3)$.

The group of <u>holonomie</u> Γ indicates how the coefficients of a harmonic quadratic form containing l + 1 variables are transformed by the orthogonal substitution effected on the variables.

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* Annales de L'Ecole Norm. Sup. tome 31. 1914.

The irreducible spaces of type (A.I) are of negative curvatures and are the representative spaces of the positive definite quadratic forms. The representation is done in such a way that two quadratic forms which differ only by a factor are represented by the same point. In this case the group G indicates how these forms are transformed into themselves by the linear substitution effected on the variables.

I shall give some informations about these spaces in the following section of this work.

S E C T I O N II.

Geometrical study of the spaces of Riemann belonging to the class A.I.of Cartan.

C H A P T E R I.

Geometrical interpretation of ds^2.

Construction of ds^2.

23. Consider, in a plane, the conics determined by their equations in honogeneous coordinates

$$A \equiv \sum_{i,j}^{1,\dots,3} A_{ij} x_i x_j = 0.$$

Consider a space E_5 of which the points - given by the honogeneous coordinates A_{ij}($i,j = 1,\dots,3$) - satisfy the condition

$$\| A_{ij} \| > 0,$$

the A_{ij} being subjected to some relation of homogeneity.

Being given in this space two infinitely near points A_{ij} and $A_{ij} + dA_{ij}$, we express the distance between these two points by means of the expression

$$d\sigma^2 = \lambda \frac{\Delta_2}{\Delta} + \mu \left(\frac{\Delta_1}{\Delta} \right)^2$$

where

$\Delta = \| A_{ij} \|$; λ and μ are numerical constants,

$$\Delta_2 = \begin{vmatrix} A_{11} & dA_{21} & dA_{31} \\ A_{12} & dA_{22} & dA_{32} \\ A_{13} & dA_{23} & dA_{33} \end{vmatrix} + \begin{vmatrix} dA_{11} & A_{21} & dA_{31} \\ dA_{12} & A_{22} & dA_{32} \\ dA_{13} & A_{23} & dA_{33} \end{vmatrix} + \begin{vmatrix} dA_{11} & dA_{21} & A_{31} \\ dA_{12} & dA_{22} & A_{32} \\ dA_{13} & dA_{23} & A_{33} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} A_{11} & A_{21} & dA_{31} \\ A_{12} & A_{22} & dA_{32} \\ A_{13} & A_{23} & dA_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & dA_{21} & A_{31} \\ A_{12} & dA_{22} & A_{32} \\ A_{13} & dA_{23} & A_{33} \end{vmatrix} + \begin{vmatrix} dA_{11} & A_{21} & A_{31} \\ dA_{12} & A_{22} & A_{32} \\ dA_{13} & A_{23} & A_{33} \end{vmatrix}$$

It is easy to verify that the ds^2 is invariant with respect to the linear transformations effected on the variables x_1, x_2, x_3.

We obtain, therefore, a space of Riemann possessing the property that <u>it admits a group of motions</u>.

24. Consider now in a space of n dimensions the surfaces of second degree - we call them hyperquadrics

$$A \equiv \sum_{i,j}^{1,\dots,n} A_{ij}\, x_i\, x_j = 0, \quad \|A_{ij}\| > 0 .$$

It is possible to represent each of them in a space of $N = \frac{1}{2}n\,(n+1)-1$ dimensions E_N by the same process as we have done in the particular case of $n = 3$. The ds can, therefore, be taken in the most simple form

$$ds^2 = \lambda \frac{\Delta_2}{\Delta} + \mu \left(\frac{\Delta_1}{\Delta}\right)^2$$

where we have $\quad \Delta = \|A_{ij}\|$,

$$\Delta_2 = \int \begin{vmatrix} A_{11} & A_{21} & \cdots & A_{n-2\,1} & dA_{n-1\,1} & dA_{n1} \\ \vdots & \vdots & & & & \\ \vdots & \vdots & & & & \\ A_{1n} & A_{2n} & \cdots & A_{n-2\,n} & dA_{n-1\,n} & dA_{nn} \end{vmatrix},$$

$$\Delta_1 = \sum \begin{vmatrix} A_{11} & A_{21} & \cdots & A_{n-1\,1} & dA_{n1} \\ \vdots & \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{n-1\,n} & dA_{nn} \end{vmatrix}$$

25. Let there be two infinitely near hyperquadrics: (A) and (A + dA). Consider a particular case when the points of intersection of these two surfaces are simple; that is to say, that there is <u>not</u> any <u>contact</u> between the two surfaces.

In this case it is permissible to choose a triangle of reference self-conjugate with regard to both surfaces such that the coordinates of these two surfaces in the space E_N be

$$A_{11}\ A_{22} \cdots\cdots A_{nn}\quad 0 \quad 0 \quad 0 \ldots 0$$

$$A_{11}\quad dA_{11} \cdots\cdots A_{nn}\quad dA_{nn}\quad 0 \cdots\cdots 0$$

The ds^2 will have, then, a very simple form

$$\Delta = \prod_1^n A_{ii}$$

$$\Delta_1 = \sum dA_{11} \cdot A_{22} \cdots A_{nn}$$

$$\Delta_2 = \sum dA_{11}\, dA_{22} \cdots A_{nn}$$

Therefore

$$\frac{\Delta_1}{\Delta} = \sum \frac{dA_{11}}{A_{11}} \,,$$

$$\frac{\Delta_2}{\Delta} = \sum \frac{dA_{11}}{A_{11}} \frac{dA_{22}}{A_{22}} \,,$$

and

$$ds^2 = \mu \sum_i^{1,\ldots,n} \left(\frac{dA_{ii}}{A_{ii}}\right)^2 + (2\mu + \lambda) \sum_{i,j}^{1,\ldots,n} \frac{dA_{ii}}{A_{ii}} \frac{dA_{jj}}{A_{jj}} \,, \qquad i \neq j$$

If for the relation of homogeneity we take

$$\Delta = \|A_{ij}\| = 1$$

we get
$$\sum_{i}^{1,\dots,n} \frac{dA_{ii}}{A_{ii}} = 0,$$
and the ds^2 takes the following form

$$ds^2 = -\sum_{i,j}^{1,\dots,n} \frac{dA_{ii}}{A_{ii}} \frac{dA_{jj}}{A_{jj}} \quad , \; i \neq j$$

where we suppose $\lambda = -1$.

Let us take, in particular, the case when $n = 3$

$$ds^2 = -\left[\frac{dA_{11}}{A_{11}}\frac{dA_{22}}{A_{22}} + \frac{dA_{33}}{A_{33}}\left(\frac{dA_{11}}{A_{11}} + \frac{dA_{22}}{A_{22}}\right)\right]$$
$$= \left(\frac{dA_{11}}{A_{11}} + \frac{dA_{22}}{A_{22}}\right)^2 - \frac{dA_{11}}{A_{11}}\frac{dA_{22}}{A_{22}} .$$

or,

$$ds^2 = \left(\frac{dA_{11}}{A_{11}}\right)^2 + \left(\frac{dA_{22}}{A_{22}}\right)^2 + \frac{dA_{11}}{A_{11}}\frac{dA_{22}}{A_{22}} .$$

<u>Case I. when the conics have no contact.</u>

26. Consider, now, a triangle of reference self-conjugate with regard to the two conics (A) and (A+dA) having <u>no contact.</u> Let S_1, S_2, S_3 be its vertices. Two points, belonging respectively to each of our two conics, lie on the straight line S_1S_2 between the points S_1 and S_2.

Let us take

$$B_{ii} = A_{ii} + dA_{ii}$$

The equations of the two conics become

$$\begin{cases} A_{11}x^2 + A_{22}y^2 + A_{33}z^2 = 0, \\ B_{11}x^2 + B_{22}y^2 + B_{33}z^2 = 0. \end{cases}$$

We can suppose that the coordinates of S_3 are

$$(0, 0, 1)$$

and the equations of the straight lines S_1S_2, S_1S_3, S_2S_3 are respectively

$$z = 0, \quad y = 0, \quad x = 0.$$

Then the points of intersection of the conic (A) with the straight line S_1S_2 is given by the equation

$$A_{11}x^2 + A_{22}y^2 = 0$$

i.e.,

$$p_A = \frac{x}{y} = \pm i\sqrt{\frac{A_{22}}{A_{11}}}$$

Similarly the points of intersection of the conic (B) would be

$$p_B = \frac{x}{y} = \pm i\sqrt{\frac{B_{22}}{B_{11}}}.$$

Let us consider, now, the cross-ratio of four points

$$R_{12} \equiv (0, +\infty, p_A, p_B) = \frac{p_A}{p_B} = \sqrt{\frac{A_{22}}{A_{11}} : \frac{B_{22}}{B_{11}}}$$

$$= \frac{1}{2}\left(\frac{dA_{22}}{A_{22}} - \frac{dA_{11}}{A_{11}}\right)$$

Similarly

$$R_{13} = \frac{1}{2}\left(\frac{dA_{33}}{A_{33}} - \frac{dA_{11}}{A_{11}}\right),$$

$$R_{23} = \frac{1}{2}\left(\frac{dA_{33}}{A_{33}} - \frac{dA_{22}}{A_{22}}\right).$$

We shall recall, moreover, that

$$\sum_{i}^{1,\dots,3} \frac{dA_{ii}}{A_{ii}} = 0.$$

Therefore,

$$R_{12}^2 + R_{13}^2 + R_{23}^2 = \frac{3}{2}\left[\left(\frac{dA_{11}}{A_{11}}\right)^2 + \left(\frac{dA_{22}}{A_{22}}\right)^2 + \frac{dA_{11}}{A_{11}}\frac{dA_{22}}{A_{22}}\right]$$

i.e.,

$$d\sigma^2 = \frac{2}{3}\left[R_{12}^2 + R_{13}^2 + R_{23}^2\right].$$

From this we deduce a result true only when these two concics have <u>no contact.</u>

The ds is a quadratic form with constant coefficients of R_{12}, R_{13}, R_{23}.

The cross-ratios (R_{12}, R_{13}, R_{23}) having an evident geometrical significance, we have, in this particular case, interpreted the ds^2 geometrically.

27. This result can be directly generalized to the case when n is not necessarily equal to 3 but any number whatsoever.

We have, in this case

$$R_{ij} = \frac{1}{2}\left(\frac{dA_{jj}}{A_{jj}} - \frac{dA_{ii}}{A_{ii}}\right) = \left(S_i S_j p_A p_B\right)$$

along with the condition

$$\sum_{i}^{1,\dots,n} \frac{dA_{ii}}{A_{ii}} = 0 .$$

The ds^2 expressing the distance between the points A_{ij} and $A_{ij}+dA_{ij}$ of two infinitely near hyperquadrics (A) and (A+dA) having <u>no contact</u> can be expressed as a quadratic form with constant coefficients of R_{ij} and hence has a geometrical significance.

28. The ds^2 of our space, in this particular case, can also be expressed as a function of the values of the cross-ratio~~n~~ of four fixed points on the conics.

Consider the conics through the four points P_1, P_2, P_3, P_4. Taking the harmonic triangle of this quadrangle as triangle of reference, the coordinates of the four points will be ($\pm p$, $\pm q$, $\pm r$), and the equations of the conics are

$$A \equiv A_{11}x^2 + A_{22}y^2 + A_{33}z^2 = 0,$$

$$B \equiv B_{11}x^2 + B_{22}y^2 + B_{33}z^2 = 0.$$

The values of the cross-ratio of the four points taken in different orders are respectively

$$\left\{ -\frac{A_{22}}{A_{11}}\frac{q^2}{p^2}, -\frac{A_{33}}{A_{22}}\frac{r^2}{q^2}, -\frac{A_{11}}{A_{33}}\frac{p^2}{r^2} \right\} \longrightarrow \quad \text{on A}$$

and $$\left\{ -\frac{B_{22}}{B_{11}}\frac{q^2}{p^2}, -\frac{B_{33}}{B_{22}}\frac{r^2}{q^2}, -\frac{B_{11}}{B_{33}}\frac{p^2}{r^2} \right\} \longrightarrow \quad \text{on B}$$

Taking the ratio of these values of the cross-ratio we have, neglecting the terms of higher degree as well as constants

$$R^*_{21} = \frac{B_{22}}{B_{11}} : \frac{A_{22}}{A_{11}} = \frac{dA_{22}}{A_{22}} - \frac{dA_{11}}{A_{11}},$$

$$R^*_{32} = \frac{B_{33}}{B_{22}} : \frac{A_{33}}{A_{22}} = \frac{dA_{33}}{A_{33}} - \frac{dA_{22}}{A_{22}},$$

$$R^*_{13} = \frac{B_{11}}{B_{33}} : \frac{A_{11}}{A_{33}} = \frac{dA_{11}}{A_{11}} - \frac{dA_{33}}{A_{33}}.$$

Let us subject A_{ij}, in the two cases, to the conditions of homogeneity

$$\Delta = 1 \; ; \; \bar{\Delta} = 1$$

For the first surface we shall have

$$\Delta \equiv A_{1n}^2 \Delta' = 1 \, , \; \Delta' = \|A_{ij}\| \quad i,j = 2 \longrightarrow n-1$$

For the second surface contained in the hyperplane $x_n = \lambda x_1$

$$\bar{\Delta} \equiv 2\lambda A_{1n} \Delta' = 1 .$$

we deduce from them

$$2\lambda A_{1n} = A_{1n}^2$$

$$\bar{\Delta} = A_{1n}^2 \Delta'$$

Therefore Δ and $\bar{\Delta}$ are identical except for their signs, and the expressions of their ds^2 are also identical except for their signs.

Consider the case when the surfaces of second degree obtained by the intersection of the hyperplane $x_n - \lambda x_1 = 0$ with the surface under consideration have no contact. The preceding paragraphs provide a geometrical interpretation of their ds^2.

Let us suppose they have two contacts at two points. Then we shall make with these surfaces the same reasoning as given before, i.e., by intersecting them by any hyperplane passing by the intersection of two tangents which are common to them. If the surfaces resulting from the intersection have no contact they give a geometrical interpretation of ds^2 , otherwise we shall repeat the same operation.

Case III. when the surfaces have a single contact.

30. Let us study the case when two surfaces of second degree have only a single contact at a given point S_n, the coordinates of point S_1 being

$$(1,0,\ldots\ldots,0)$$

and the equation of the tangent plane being

$$x_n = 0.$$

The surface whose equation is

$$A \equiv \sum_{i,j}^{1,\ldots,n} A_{ij}\, x_i x_j = 0$$

will evidently satisfy the conditions

$$A_{1i} = 0 \quad i < n$$

and we see, as in the preceding paragraph, that

$$\Delta = A_{1n}^2 \, \Delta'$$

Δ' having the same form as given in the preceding case.

Therefore, the ds^2 is the same whatsoever be the value of A_{ni} ($i \neq 1$).

We can, therefore, give some arbitrary values to these A_{ni}. In particular let us give them the value zero. This means that we come back to consider among the surfaces having the same ds^2, two surfaces having no more one but two contacts.

Thus there are an infinity of surfaces having the same ds^2, and a point of contact. They are those for which the A_{ij} have the same values with the exception of A_{ni} ($i \neq 1$).

When these last quantities tend to zero, the corresponding surface tend to a limiting position for which there are <u>two points of contact</u> with the first given surface.

Thus the ds^2 of two surfaces having only one point of contact are brought back to those two surfaces having two points of contact.

<u>Case IV. when the surfaces have osculating contact.</u>

31. The cubic equation

$$\Delta\lambda^3 + \Theta\lambda^2 + \Theta'\lambda + \Delta' = 0$$

determines three values of λ for which the conic

$$\lambda S + S' = 0$$

breaks up into a pair of straight lines.

If the three roots of the cubic in λ are equal, three of the points of intersection coincide and we have a three-point contact, the conditions for the three-point contact being

$$\frac{3\Delta}{\Theta} = \frac{\Theta}{\Theta'} = \frac{\Theta'}{3\Delta'}$$

or,

$$\frac{3\Delta}{3\Delta + \Delta_1} = \frac{3\Delta + \Delta_1}{3\Delta + 2\Delta_1 + \Delta_2} \qquad \text{(i)}$$

and

$$\frac{3\Delta}{3\Delta + \Delta_1} = \frac{3\Delta + 2\Delta_1 + \Delta_2}{3(\Delta + \Delta_1 + \Delta_2 + \Delta_3)} \qquad \text{(ii)}$$

where $\Delta, \Delta_1, \Delta_2$ have the same values as given in paragraph 23, and

$$\Delta_3 = \begin{vmatrix} dA_{11} & dA_{21} & dA_{31} \\ dA_{12} & dA_{22} & dA_{32} \\ dA_{13} & dA_{23} & dA_{33} \end{vmatrix}.$$

The conditions (i) and (ii) are simplified to

$$\frac{3\Delta_2}{\Delta} - \frac{\Delta_1^2}{\Delta^2} = 0 \text{ ———— } (i)$$

and

$$\frac{9\Delta_3}{\Delta} - \frac{\Delta_1\Delta_2}{\Delta^2} = 0 \text{ ———— } (ii)$$

If for the relation of homogeneity we take

$$\Delta = 1,$$

then
$$\frac{\Delta'}{\Delta} = 0$$

and ds^2 reduces to $ds^2 = \lambda \frac{\Delta_2}{\Delta}$

which is zero in virtue of the condition (i)

Thus the ds^2 of our space, in this case, is zero.

Similarity with the spaces of Cayley.

32. Let us consider, now, the ds^2 in its general form

$$ds^2 = \lambda \frac{\Delta_2}{\Delta} + \mu \left(\frac{\Delta_1}{\Delta}\right)^2$$

Consider, in the space E_N, a point of which the coordinates are A_{ij}.

We replace, in the expression of ds^2, the quantities dA_{ij} by any quantities B_{ij}.

Let

$$\phi(A_{ij}, B_{ij})$$

be the expression thus obtained. It is of the second degree and homogeneous in the B_{ij}.

Let us take

$$\phi(A_{ij}; B_{ij}) = \text{constant.}$$

For

$B_{ij} = A_{ij}$ we have $\Delta_2 = 3\Delta = \Delta_1$

Therefore, $\phi(A_{ij}; A_{ij}) = 3\lambda + 9\mu$.

and the constant is equal to $3\lambda + 9\mu$.

We can, therefore, consider the points of the space E_N determined by the system of coordinates (B_{ij}). These coordinates are homogeneous and are subjected to the relation of homogeneity

$$\phi = 3\lambda + 9\mu$$

Consider, then, in this space, the surface of the second degree

$$\phi(A_{ij}; B_{ij}) = 0,$$

<u>the A_{ij} being considered as some constants</u>, and the ds^2 defined by the quadratic form

$$ds^2 = \phi(A_{ij}; dB_{ij})$$

It follows from the above consideration that the ds^2 of our space of Riemann is identical, at the point A_{ij}, with that of a space with constant curvature of the first kind referred to the absolute

$$\phi(A_{ij}; B_{ij}) = 0$$

and susceptible of geometrical interpretation, with respect to this absolute, by means of the cross-ratio of four points: (A_{ij}), $(A_{ij} + dA_{ij})$ and two points of intersection of the straight line joining these two points with the surface of the second degree $\phi = 0$.

<u>Thus E_N can be considered as constituting small bits of the classical non-euclidean spaces of Cayley.</u>

C H A P T E R II.

Determination of the totally geodesic varieties of the space E_N.

Totally geodesic varieties.

33. If from a given point A of a space of Riemann we draw different geodesics which are tangent, at this point, to the same p-dimensional element plane, the surface generated by these geodesics is called a p-dimensional geodesic variety at the point A. If the variety is geodesic at each of its points, it is said to be totally geodesic variety of the space of Riemann. The totally geodesic variety possesses a characteristic property namely " any geodesic which is tangent to the variety is contained entirely into it."

In the following paragraphs I propose to determine the totally geodesic varieties of our space E_N.

Representative space of the group G.

34. Let us consider the general projective group in the space E_n. It is defined by the transformations

$$(G)\qquad \boxed{\begin{aligned} Z_{\alpha\beta} &= X_{\alpha\beta} + X_{\beta\alpha} \\ Y_{\alpha\beta} &= X_{\alpha\beta} - X_{\beta\alpha} \\ X_{\alpha} &= X_{\alpha\alpha} \end{aligned}}$$

where

$$X_{\alpha\beta} = x_\alpha \frac{\partial}{\partial x_\beta}$$

In Journal de mathematique - 1927, E. Cartan has shown how with such a group we can make to correspond a variety of

$$r = n^2 - 1$$

dimensions of affine connection without <u>torsion</u>.

The group G consists of r parameters: $a_1, \ldots, a_r$; we shall regard the parameters of this group as the coordinates of a point (a) in a space of r dimensions E_r which is called the group-space.

The properties of group permit us to introduce some geometrical notions in this continuum.

Being given two points (a) and (b) of this space of group we shall call the aggregate of these two points as vector - the points (a) and (b) being respectively the origin and the extremity of this vector. The vector will be denoted by the notation $\vec{ab}$.

With each point (a) is evidently associated a transformation T_a of the group. We define two kinds of equipollence :

Equipollence of first kind $[(1)]$

$$\vec{ab} \overset{(1)}{=} \vec{a'b'}$$

if $$T_b T_a^{-1} = T_{b'} T_{a'}^{-1} ;$$

Equipollence of second kind $[(2)]$

$$\vec{ab} \overset{(2)}{=} \vec{a'b'}$$

if $$T_a^{-1} T_b = T_{a'}^{-1} T_{b'}$$

Without any loss of generality we can consider the origin of our space E_r chosen in such a way that it coincides with the point (a_0) corresponding to the identity transformation of the group G.

Consider, moreover, a point, defined by the quantities $da_1, \ldots\ldots, da_r$, infinitely near the origin.

The transformation T_{0+da} is an infinitesimal transformation which is deduced evidently from r infinitesimal transformations of bases $X_1f, \ldots\ldots, X_rf$ according to the theory of S.Lie.

The vector $\overrightarrow{aa'}$ joining a point (a) to an infinitely near point (a') - $a'_i = a_i + da_i$ - will be equipollent to the vector $\overrightarrow{Oda}$ if we have

$$T_{0+da} T_0^{-1} = T_{a'} T_a^{-1} = T_{a+da} T_a^{-1}$$

Now

$$T_{a+da} T_a^{-1} = \tilde{\omega}^1 X_1 f + \cdots + \tilde{\omega}^r X_r f$$

where $\tilde{\omega}^i$ are the forms of Pfaff of the first order in da_i.

If we consider, at each point of E_r, the cartesian <u>repère</u> obtained by drawing from this point the vectors equipollent to the vectors forming the cartesian repère at the origin, the $\tilde{\omega}^i$ will be the coordinates of the point $a' = a + da$ referred to the cartesian <u>repère</u> at the point (a).

Similarly we can attach, at each point of E_r, a cartesian <u>repère</u> deduced by the equipollence of the second kind from the <u>repère</u> at the origin. The coordinates of the point $a' = a + da$ infinitely near the point (a) is given by ω^i, where ω^i are defined by

$$T_a^{-1} T_{a+da} = \omega^1 X_1 f + \cdots + \omega^r X_r f$$

Moreover, we have the following equations of structure.

$$(X_j X_k) = \sum_\rho C_{jk}^{\cdot\cdot\rho} X_\rho f,$$

$$(\tilde{\omega}^\rho)' = -\sum_{(jk)} C_{jk}^{\cdot\cdot\rho} [\tilde{\omega}^j \tilde{\omega}^k],$$

$$(\omega^\rho)' = \sum_{(jk)} C_{jk}^{\cdot\cdot\rho} [\omega^j \omega^k].$$

We have thus defined two affine connections without curvature which permit us to define a third affine connection without tersion defined by

(1). the forms $\tilde{\omega}^i$

(2). the forms $\omega_i^j = \frac{1}{2} \tilde{\omega}_i^j$

where $$\tilde{\omega}_i^j = \sum_k C_{ki}^{\cdot\cdot j} \omega^k.$$

The curvature of the space is, then, defined by the forms

$$\Omega_i^j = (\omega_i^j)' - \sum_k [\omega_i^k \omega_k^j]$$
$$= \frac{1}{2} (\tilde{\omega}_i^j)' - \frac{1}{4} \sum_k [\tilde{\omega}_i^k \tilde{\omega}_k^j]$$

The Riemann-Christoffel tensor is given, here, by

$$R_{i\cdot\alpha\beta}^{j} = \frac{1}{4} \sum_k C_{\alpha\beta}^{\cdot\cdot k} C_{ki}^{\cdot\cdot j}$$

We can, now, define a symbol Uf by

$$Uf = \sum_i \tilde{\omega}^i X_i f$$

<u>Totally geodesic varieties of E_r.</u>

35. We enunciate a theorem, due to E. Cartan, which he has obtained for the determination of the totally geodesic varieties of the representative spaces of the group G.

<u>Theorem</u>: At any point of a totally geodesic variety V_p of p dimensions the symbols $U_1 f, \ldots, U_p f$ of p independent directions of this variety possess the property that the infinitesimal transformations

$$((U_j U_k)\, U_i)$$

depend linearly (with constant coefficients) on the transformations

$$U_1, \ldots\ldots, U_p$$

This theorem has a first immediate application:

The totally geodesic varieties of E_r which are the representative varieties of a sub-group and are not ~~not~~ contained in any representative variety of a sub-group are the spaces of Riemann of the type indicated and studied by E. Cartan in the memoir "Bull. Soc., Math. de France tome 55. 1927."

Now in the case of the group G it is easy to find a such totally geodesic variety.

we have

$$(Y_{\alpha\beta} Y_{\beta\gamma}) = Y_{\alpha\gamma}\ ;\ (Z_{\alpha\beta} Z_{\beta\gamma}) = Y_{\alpha\gamma}\ ;\ (Y_{\alpha\beta} Z_{\beta\gamma}) = Z_{\alpha\gamma}\ ;$$

$$(X_\alpha Y_{\alpha\beta}) = Z_{\alpha\beta}\ ;\ (X_\alpha Z_{\alpha\beta}) = Y_{\alpha\beta}\ ;\ (Y_{\alpha\beta} Z_{\alpha\beta}) = 2(X_\alpha - X_\beta)$$

The other brackets, where $\alpha, \beta, \gamma, \delta$ are all different, are zero.

Whence we have

$$(Z_{\alpha\gamma}(Z_{\alpha\beta}Z_{\beta\gamma})) = 2(X_\gamma - X_\alpha)$$

$$(Z_{\gamma\rho}(Z_{\alpha\beta}Z_{\beta\gamma})) = -Z_{\alpha\rho}$$

$$(Z_{\beta\rho}(X_\alpha Z_{\alpha\beta})) = -Z_{\alpha\rho}$$

$$(Z_{\beta\alpha}(X_\alpha Z_{\alpha\beta})) = 2(X_\beta - X_\alpha)$$

$$(X_\alpha(X_\alpha Z_{\alpha\beta})) = Z_{\alpha\beta}$$

$$(X_\beta(X_\alpha Z_{\alpha\beta})) = -Z_{\alpha\beta}$$

$$(X_\alpha(Z_{\alpha\gamma}Z_{\gamma\beta})) = Z_{\alpha\beta}$$

It follows from this that, in the group-space E_r of affine connection without torsion, the surfaces tangent, at each point, in the directions

$$X_\alpha - X_1 ; \; Z_{\alpha\beta}$$

where α, β are any number, are totally geodesic varieties which considered themselves are the spaces of Riemann of the type under consideration.

<u>Totally geodesic varieties of our space</u> E_N.

36. Consider a point P of this variety. Let us correspond to it any surface (A) of the second degree of the space E_n. As we can refer this surface to any polyhydra of reference in the space E_n, it will be

proper to choose those with respect to which the equation of the surface (A) takes the following form:

$$\sum_i x_i^2 = 0 \qquad \left| \begin{array}{l} A_{ii} = 1 \\ A_{ij} = 0 \end{array} \right. \quad (\Delta = 1).$$

Let us associate to any point P' - infinitely near from P - the surface (A') in the space E_n. This is obtained by applying the transformation

$$T_{P'} T_P^{-1}$$

to the surface (A).

We are going to justify that this is possible because the number of dimensions of the proposed variety is $N = \frac{1}{2} n(n+1) - 1$. Also to any two distinct points of the variety there correspond two distinct surfaces.

This means that we can establish a correspondence between the r parameters of the group G and the coefficients A_{ij} where we consider A_{ij} and A_{ji} as two distinct variables. The variety E_N is, then, generated by the transformations of the points P.

37. Thus we realize a point correspondence between the variety from which we started and the space E_N. This correspondence is evidently isometric. Thus the two spaces admit the same group of motions. Any application of a space on the other, possessing this property, defines an isometric correspondence.

38. It is now easy to find certain number of totally geodesic varieties of our space E_N and their geometrical interpretation.

First of all any variety of $p \leqslant n-1$ dimensions generated by the directions

$$X_\alpha - X_1$$

These are the surfaces having the geometry of the euclidean spaces.

If we take $\frac{dA_{ii}}{A_{ii}} = d\xi_i$

the equations of the geodesic contained in these surfaces will have the form : $\xi_i = \lambda_i t + \mu_i$.

It is easy to see that these surfaces are geometrically represented by the aggregate of the surfaces (A) admitting the same self-conjugate (auto-polar) polyhydra of reference.

We consider, secondly, the surfaces generated by the directions

$$X_2 - X_1,\ X_3 - X_1,\ Z_{12},\ Z_{13},\ Z_{23}$$

Let us start from the surface (A) of the second degree

$$\sum_i x_i^2 = 0$$

and apply to it the transformation

$$e_2(X_2 - X_1) + e_3(X_3 - X_1) + e_{12} Z_{12} + e_{13} Z_{13} + e_{23} Z_{23}.$$

That is to say

$$e_2\left(x_2\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_1}\right) + e_3\left(x_3\frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_1}\right) + e_{12}\left(x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_1}\right)$$
$$+ e_{13}\left(x_1\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_1}\right) + e_{23}\left(x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_2}\right)$$

The increase thus obtained is evidently

$$e_2(2x_2^2-2x_1^2)+e_3(2x_3^2-2x_1^2)+e_{12}\cdot 4x_1x_2+e_{13}\cdot 4x_1x_3+e_{23}\cdot 4x_2x_3$$

The equation of the surface (A') will be

$$x_1^2\left[1-2(e_2+e_3)\right]+x_2^2(1+2e_2)+x_3^2(1+2e_3)+x_4^2+\cdots+x_n^2+4e_{12}x_1x_2+4e_{13}x_1x_3+4e_{23}x_2x_3=0$$

Consider a polyhydra of reference $S_1, S_2, \ldots\ldots, S_n$ self-conjugate to the surface (A). It is easy to see that the vertices $S_4, S_5, \ldots\ldots, S_n$ admit the same conjugate hyperplane with respect to the aggregate of the surfaces (A').

The totally geodesic surface having the greatest number of dimensions will be, therefore, that when for instance the vertex $(0,0,\ldots\ldots,1)$ will admit the same conjugate plane with respect to the aggregate of surfaces (A').

The equation of the surface (A') is therefore

$$\lambda x_\beta^2+\sum_{i,j\neq\beta}A_{ij}\,x_i x_j=0\ ,\ \beta=2,\ldots,n$$

These surfaces are obtained by the following directions for $\beta = n$

$$X_2-X_1\,,\ \ldots\,,\ X_n-X_1\,,\ Z_{\alpha\beta}\quad \left.{}^{\alpha}_{\beta}\right\}=1,\ldots,n-1\,.$$

They are, therefore, of

$$n+\frac{(n-1)(n-2)}{2}-1=\frac{1}{2}n(n-1)$$ dimensions.

39. In the case of n = 3, for example, the maximum number of dimensions of a totally geodesic surface is 3.

They will be generated by the transformations

(i) X_2-X_1, X_3-X_1, Z_{12}

(ii) X_3-X_1, X_2-X_1, Z_{13}

(iii) X_2-X_1, X_3-X_2, Z_{23}

They are represented geometrically by the aggregate of concentric circles. We see that such a surface contains an infinity of totally geodesic surfaces of 2 dimensions, having a common self-conjugate triangle. Also we can easily verify their properties of being such that through two points there passes only one geodesic.

THE END.

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One parameter continuous groups of Deformations.

By

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Introduction: Taking a one parameter transformation of a Lie group generated by the infinitesimal transformation.

$$(1) \qquad D = u^r \frac{\partial}{\partial x^r}$$

in a space whether Riemannian or of affine connection, Kosambi was the first to suggest* the idea that if the group transforms the points and the coordinates, not the tensors, the new components of these tensors are determined by a suitable operator associated with the given Lie group.

This tensor operator I show to be actually S defined by

$$(2) \qquad ST^{\cdot\cdot}_{\cdot\cdot} = u^r T^{\cdot\cdot}_{\cdot\cdot|r} + u^{\cdot}_{|\cdot} T^{\cdot\cdot}_{r\cdot} + \cdots + 2u^r T^{\cdot\cdot}_{m\cdot} \Omega^{m}_{\cdot r} + \cdots - u^{\cdot}_{|r} T^{r\cdot}_{\cdot\cdot} - \cdots - 2u^r T^{m\cdot}_{\cdot\cdot} \Omega^{\cdot}_{mr} - \cdots$$

and

$$(3) \qquad \overline{T}^{ij\cdot\cdot}_{hk\cdot\cdot} = e^{\varepsilon S} T^{ij\cdot\cdot}_{hk\cdot\cdot}$$

using the well known notation of H. Poincaré.

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* D. D. Kosambi: " Continuous groups and two theorems of Euler" (Math. Student, II, 1934, 94-100)*

Here $T^{ij\cdot\cdot}_{hk\cdot\cdot}$ is a tensor of the space, and $\bar{T}^{ij\cdot\cdot}_{hk\cdot\cdot}$ is the corresponding new tensor of the space obtained by the operator S as shown in (3).

Kosambi's original programme was to investigate the main possibilities that immediately suggest themselves when the equations of Killing

$$u_{i/j} + u_{j/i} = 0$$

are replaced by the vanishing of the S-variation of other fundamental geometric objects, i.e., the Riemann-Christoffel curvature tensor, Weyl conformal and projective tensors etc. A greater generalization is not possible as his notes contain a fairly simple result that under the transformation inverse of $\bar{x}^i = e^{\varepsilon D} x^i$ * we have

(4) $$\bar{g}_{pq}(\bar{x}) \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \equiv g_{ij}(x),$$

where

$$\bar{g}_{ij}(x) = e^{\varepsilon S} g_{ij}.$$

/ Also his unpublished notes dated the same year.
I express my thanks to Prof. D. D. Kosambi for his guidance and suggestions throughout the course of this investigation.

* Questions of convergence are ignored throughout, as we assume the relevant series always convergent for all sufficiently small values of the parameter and some suitably restricted region of the space.

Thus the space as a whole does not change merely by Lie transformation which can always be visualised as a change of coordinates.

I show, using the S-operator, that isotropic Riemann spaces remain isotropic with the same scalar curvature as would be expected from the above. This is , presumably, the true significance of the well-known results that an isotropic space admits isometric correspondence with ½n (n+1) parameters, and also ½ n (n+1) independent solutions of the equations of Killing.

The emphasis of our investigation is entirely different from that given in standard text books on the theory of continuous groups of transformations, where we see what are the conditions that a space admit a given group. Here we are concerned with both the original and the deformed spaces, their invariants and the relations that subsist between them. Incidentally we have:

" If a tensor T remains invariant under the infinitesimal transformation of a group it remains invariant under the finite transformation of the group."

I. <u>The S-operator:</u>

The original transformation, represented by the operator $D = u^{r}\frac{\partial}{\partial x^{r}}$, applies only to point functions as such. Its first extension, represented by

$$M = u^{r}\frac{\partial}{\partial x^{r}} + \dot{u}^{r}\frac{\partial}{\partial \dot{x}^{r}}$$

is applied to the metric $F = g_{ij}\dot{x}^{i}\dot{x}^{j}$ and we have

$$\bar{F} = e^{\varepsilon M} F$$

where

$$MF = u^{r}F_{,r} + \dot{u}^{r}F_{;r}$$

or, intensorial form

$$MF = u^{r}F_{|r} + u^{r}_{|k}\dot{x}^{k}F_{;r}$$

Here comma, semi-colon and solidus denote differentiation with respect to x^{i}, $\dot{x}^{i}$ and Kosambi's covariant differentiation respectively.

On account of the point of view adopted, the following lemmas are easily proved:

Lemma I:

$$S dx^{i} = 0$$

Lemma II: $$M(A_i dx^{i}) = (SA_i)\, dx^{i}$$

where

$$SA_i = u^{r}A_{i|r} + u^{r}_{|i}A_r + 2u^{r}A_m\Omega^{m}_{\cdot ir}$$

Lemma III : $\mathcal{S}B^i = u^r B^i_{|r} - u^i_{|r} B^r - 2u^r B^m \Omega^i_{mr}$

For $\mathcal{S}(A_i B^i) = (\mathcal{S}A_i)B^i + A_i(\mathcal{S}B^i) = u^r \frac{\partial}{\partial x^r}(A_i B^i)$

Lemma IV:

$$\mathcal{S}T^{\cdot\cdot}_{\cdot\cdot} = u^r T^{\cdot\cdot}_{\cdot\cdot|r} + u^r_{|\cdot} T^{\cdot\cdot}_{r\cdot} + \cdots\cdots + 2u^r T^{\cdot\cdot}_{m\cdot}\, \Omega^m_{\cdot r} + \cdots$$
$$- u^{\cdot}_{|r} T^{r\cdot}_{\cdot\cdot} - \cdots - 2u^r T^{m\cdot}_{\cdot\cdot}\, \Omega^{\cdot}_{mr} - \cdots$$

This is easily proved from the identity

$$S\,(T^{ij\cdot\cdot}_{hk\cdot\cdot} A^h B^k V_i W_j \cdots\cdots) = u^r \frac{\partial}{\partial x^r}(\text{ the same}).$$

Theorem I: In particular $Sg_{ij} = u_{i/j} + u_{j/i}$

Theorem II: $\overline{T}^{ij\cdot\cdot}_{hk\cdot\cdot} = e^{\varepsilon S}\, T^{ij\cdot\cdot}_{hk\cdot\cdot}$

For let a point $\overline{A}$ be the transform of a point A. Let ε be the parameter of the transformation. Then $\overline{T}^{ij}_{hk}$ are functions of ε, and even analytic in ε. Further:

$$\lim_{\varepsilon \to 0} \frac{\overline{T}^{ij\cdot\cdot}_{hk\cdot\cdot} - T^{ij\cdot\cdot}_{hk\cdot\cdot}}{\varepsilon} = \mathcal{S}T^{ij\cdot\cdot}_{hk\cdot\cdot}$$

Hence the theorem, for this formula gives simply the Taylor's series defining $\overline{T}^{ij\cdot\cdot}_{hk\cdot\cdot}$.

Theorem III: By direct calculation the new coefficients of connection $\overline{L}^{i}_{jk}$ are obtained in the following form:

$$\overline{L}^{i}_{jk} = L^{i}_{jk} + \varepsilon a^{i}_{jk} + \cdots + \frac{\varepsilon^{n}}{n!} S^{n-1} a^{i}_{jk} + \cdots$$

where

$$a^{i}_{jk} = u^{i}_{|j|k} + u^{r} L^{i}_{jrk} + 2\left(u^{r}\Omega^{i}_{jr}\right)_{|k}$$

Theorem IV: The curvature tensor of the deformed space is given by

$$\overline{L}^{i}_{jkl} = e^{\varepsilon S} L^{i}_{jkl}$$

II. I shall, now, prove the invariance of the scalar curvature, K_0, for finite transformation of an isotropic Riemannian space.

A necessary and sufficient condition that a space V_n be of constant curvature K_0 is that the components of the fundamental tensor satisfy the conditions

$$R^{i}_{jkl} = K_0\left(\delta^{i}_{k} g_{jl} - \delta^{i}_{l} g_{jk}\right).$$

Substituting this expression in

$$\overline{R}^{i}_{jkl} = e^{\varepsilon S} R^{i}_{jkl}$$

we have

$$\overline{R}^{i}_{jkl} = K_0 e^{\varepsilon S}\left(\delta^{i}_{k} g_{jl} - \delta^{i}_{l} g_{jk}\right) = K_0\left(\delta^{i}_{k} \overline{g}_{jl} - \delta^{i}_{l} \overline{g}_{jk}\right)$$

which proves the first theorem of the introduction.

The conditions for the projective and conformal transformations are respectively

$$SW^i_{jk\ell} \equiv u^r W^i_{jk\ell|r} + u^r_{|j} W^i_{rk\ell} + u^r_{|k} W^i_{jr\ell} + u^r_{|\ell} W^i_{jkr} - u^i_{|r} W^r_{jk\ell} = 0,$$

and

$$SC^i_{jk\ell} = u^r C^i_{jk\ell|r} + u^r_{|j} C^i_{rk\ell} + u^r_{|k} C^i_{jr\ell} + u^r_{|\ell} C^i_{jkr} - u^i_{|r} C^r_{jk\ell} = 0.$$

These conditions are equivalent to those given by Eisenhart* in much less handy form; but he has given one more condition associated with his derivation

$$u_m\left[R^m_{j|k} - R^m_{k|j} + g^{i\ell}\left(R^m_{kji|\ell} - R^m_{jki|\ell}\right)\right] = 0.$$

Our powerful notation shows the above expression to be identically equal to $SR_{jk} - SR_{kj}$. It is known that for Riemannian spaces $R_{ij} = R_{ji}$, so that Eisenhart's condition is identically satisfied. A direct proof can also be given by using the identities of Bianchi and Ricci.

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* L.P. Eisenhart:- " Riemannian geometry"; equations (168.13), (68.14), (69.3), (69.4).

It is clear that if the metric tensor remains invariant under the finite transformation of the group, the curvature tensor also remains invariant. For $SR^i_{jkl} = 0$ follows from $Sg_{ij} = 0$. But if $SR^i_{jkl} = 0$, then Sg_{ij} is necessarily equal to zero only for an Einstein space ($R \neq 0$, $n \geqslant 3$) and hence for isotropic spaces also.

For if in

$$SR^i_{jkl} = 0$$

we contract for i and l, we obtain

$$SR_{jk} = u^r R_{jk/r} + u^r_{/j} R_{rk} + u^r_{/k} R_{jr} = 0$$

For an Einstein space ($R \neq 0$, $n \geqslant 3$) this reduces to

$$u_{i/k} + u_{k/i} \equiv Sg_{jk} = 0$$

Hence the proposition.